Solutions to Math 103 Final

1. Prove that there exists a Schwartz function \( h : \mathbb{R} \to \mathbb{C} \) with the following property. If \( f_1 \) is any Schwartz function on \( \mathbb{R} \) with \( \text{supp} \hat{f}_1 \subset [0, 1] \) and \( f_2 \) is any Schwartz function on \( \mathbb{R} \) with \( \text{supp} \hat{f}_2 \subset [2, 3] \), then \( (f_1 + f_2) \ast h = f_1 \).

You don’t need to write an exact formula for \( h \). Just explain why the function \( h \) exists.

(This problem is related to how a radio works. Each radio station sends out a radio signal with frequency in a different range. The antennae of a radio receives a signal which is the sum of all of these contributions. To locate the signal from a single station, we need to find the part of the incoming signal in a given frequency range.)

Solution to 1: Let \( g(\omega) \) be a \( C^\infty \) smooth function with \( g(\omega) = 1 \) for \( \omega \in [0, 1] \) and with the support of \( g \) contained in \( [-1/2, 3/2] \). Let \( h \) be the inverse Fourier transform of \( g \). Then \( g = \hat{h} \). Since \( g \) is Schwartz, \( h \) is also Schwartz.

Now we consider \( F := (f_1 + f_2) \ast h \). We consider the Fourier transform:

\[
\hat{F} = (\hat{f}_1 + \hat{f}_2) \hat{h} = (\hat{f}_1 + \hat{f}_2)g.
\]

Since \( \hat{f}_1 \) is supported in \([0, 1]\), \( \hat{f}_1 g = \hat{f}_1 \). Since \( \hat{f}_2 \) is supported in \([2, 3]\), \( \hat{f}_2 g = 0 \). Therefore, \( \hat{F} = \hat{f}_1 \). Since \( f_1 \) and \( F \) are Schwartz, we get by Fourier inversion that \( F = f_1 \). In other words, \( (f_1 + f_2) \ast h = f_1 \) as desired.

2. Suppose that \( f \) is a Schwartz function on \( \mathbb{R} \). Suppose that \( \int_{\mathbb{R}} |f|^2 = 1 \) and that \( \hat{f} \) is supported in \([-1, 1]\). Prove that for any points \( x, y \in \mathbb{R} \),

\[
|f(x) - f(y)| \leq 1000|x - y|.
\]

Solution to 2: First, by the fundamental theorem of calculus, we have

\[
|f(x) - f(y)| = \left| \int_x^y f'(z)dz \right| \leq |x - y| \max_z |f'(z)|.
\]

Therefore, it suffices to prove that for all \( z \in \mathbb{R} \), \( |f'(z)| \leq 1000 \). Next we study \( f' \) using the Fourier transform. Let \( g = f' \). Then \( \hat{g} = 2\pi i \omega \hat{f} \). By Fourier inversion we get

\[
f'(z) = \int_{\mathbb{R}} 2\pi i \omega \hat{f}(\omega)e^{2\pi i \omega z} d\omega.
\]

Since \( \hat{f} \) is supported in \([-1, 1]\), we get
\[ |f'(z)| \leq 2\pi \int_{-1}^{1} |\omega| |\hat{f}(\omega)| d\omega \leq 2\pi \int_{-1}^{1} |\hat{f}(\omega)| d\omega.\]

Using Cauchy-Schwarz and then Plancherel we see that
\[ \int_{-1}^{1} |\hat{f}(\omega)| \cdot 1 d\omega \leq \left( \int_{-1}^{1} |\hat{f}(\omega)|^2 d\omega \right)^{1/2} \cdot (2)^{1/2} = 2^{1/2} \left( \int_{\mathbb{R}} |f(x)|^2 dx \right) = 2^{1/2}.\]

So for every \( z \in \mathbb{R} \), \( |f'(z)| \leq 2 \cdot 2^{1/2} \leq 100 \) as desired.

3. Suppose that \( f \in L^1(\mathbb{R}) \). Let
\[ g = f * e^{-|x|^2} = \int_{\mathbb{R}} f(y)e^{-|x-y|^2} dy.\]

Prove that
\[ \lim_{x \to +\infty} g(x) = 0.\]

Pick \( \epsilon > 0 \). We claim that there exists some \( R < \infty \) so that \( \int_{|y|>R} |f(y)|dy < \epsilon \). To prove the claim, we observe by the monotone (or Lebesgue dominated) convergence theorem that
\[ \lim_{R \to \infty} \int_{-R}^{R} |f(y)|dy = \int_{\mathbb{R}} |f(y)|dy. \]

The integral \( \int_{\mathbb{R}} |f(y)|dy \) is finite, and so
\[ \int_{|y|>R} |f(y)|dy = \left( \int_{|y|\leq R} |f(y)|dy \right) - \left( \int_{-R}^{R} |f(y)|dy \right) \to 0. \]

Now we estimate \( g(x) \) for large \( x \). In particular, for \( x > R \), we see that
\[ |g(x)| = \left| \int_{\mathbb{R}} f(y)e^{-|x-y|^2} dy \right| \leq \left| \int_{|y|\leq R} f(y)e^{-|x-y|^2} dy \right| + \left| \int_{|y|>R} f(y)e^{-|x-y|^2} dy \right|.\]

The second term on the right-hand side is bounded by \( \int_{|y|>R} |f(y)|dy < \epsilon \). To bound the first term on the right-hand side, we note that since \( |y| \leq R \) and \( x > R \), \( e^{-|x-y|^2} \leq e^{-(x-R)^2} \). Therefore, the first term is bounded by \( e^{-(x-R)^2} \int_{\mathbb{R}} |f| \). All together, if \( x > R \), we have
\[ |g(x)| \leq \epsilon + e^{-(x-R)^2} \|f\|_{L^1}. \]
If $x$ is sufficiently large, we see that $|g(x)| < 2\epsilon$. Since $\epsilon$ is arbitrary, we see that $\lim_{x \to +\infty} g(x) = 0$.

4. Suppose that $E \subset \mathbb{R}$ is a measurable set with $m(E) = 1$. Prove that there exists an open interval $I$ with

$$m(E \cap I) \geq \frac{9}{10} m(I).$$

Solution to 4: Since $m(E)$ is finite, $E$ can be well approximated by a finite union of intervals. More precisely, for any $\epsilon > 0$, there exists a finite union of intervals $F$ so that $m(E \triangle F) < \epsilon$. Any finite union of intervals can be rewritten as a finite union of disjoint intervals in a unique way. So $F$ is a finite disjoint union of intervals $I_j$. If $\epsilon < (1/100)$, then we claim that for one of these intervals $I_j$,

$$m(E \cap I_j) \geq \frac{9}{10} m(I_j).$$

Note that $m(F \cap E) \geq m(E) - m(F \triangle E) \geq 1 - \frac{1}{100} = \frac{99}{100}$. So

$$\sum_j m(E \cap I_j) = m(F \cap E) \geq \frac{99}{100}.$$  

On the other hand, $m(F) \leq m(E) + m(F \triangle E) \leq 1 + \frac{1}{100} = \frac{101}{100}$. So

$$\sum_j m(I_j) \leq \frac{101}{100}.$$  

Combining the last two equations, we see that

$$\sum_j m(E \cap I_j) \geq \frac{99}{101} \sum_j m(I_j).$$

Therefore, there must be some $j$ so that

$$m(E \cap I_j) \geq \frac{99}{101} m(I_j) \geq \frac{9}{10} m(I_j).$$

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is $C^2$ and $2\pi$ periodic. Suppose that

$$\frac{1}{2\pi} \int_0^{2\pi} f = 1$$

$$\max_x |f'(x)| \leq 9/10.$$
Let $g_k$ be $f$ convolved with itself $k$ times. In other words, $g_1 := f$ and $g_k := g_{k-1} * f$.

(Here we use convolution for $2\pi$-periodic functions: $f * g := \frac{1}{2\pi} \int_0^{2\pi} f(y)g(x-y)dy$.)

Prove that $g_{100}$ is strictly positive.

We study the Fourier series of $f$, and use it to study the Fourier series of $g_k$.

The first equation tells us that $\hat{f}(0) = 1$. We use the inequality $|f'(x)| \leq 9/10$ to bound the other Fourier coefficients. For $n \neq 0$, integrating by parts gives us

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx}dx = -\frac{1}{2\pi} \cdot \frac{1}{i} \int_0^{2\pi} f'(x)e^{-inx}dx.$$ 

Therefore,

$$|\hat{f}(n)| \leq |n|^{-1} \frac{1}{2\pi} \int_0^{2\pi} |f'(x)|dx \leq \frac{9}{10} |n|^{-1}.$$ 

Next we consider how $\hat{g}_k$ relates to $\hat{f}$. By definition $g_k = g_{k-1} * f$. Therefore, $\hat{g}_k = \hat{g}_{k-1} \hat{f}$. Since $g_1 = f$, clearly $\hat{g}_1 = \hat{f}$. Therefore, we see that $\hat{g}_k = (\hat{f})^k$. In particular, we see that

$$\hat{g}_k(0) = 1.$$ 

$$|\hat{g}_k(n)| \leq \left(\frac{9}{10}\right)^k |n|^{-k}.$$ 

If $k$ is large, then $\hat{g}_k(n)$ becomes small for all $n \neq 0$. By Fourier inversion, we can write $g_k$ in terms of its Fourier series as

$$g_k(x) = \sum_{n=-\infty}^{\infty} \hat{g}_k(n)e^{inx} = 1 + \sum_{n\neq 0, n \in \mathbb{Z}} \hat{g}_k(n)e^{inx}.$$ 

If $k$ is large then the term 1 dominates the remaining term. In fact

$$\left|\sum_{n\neq 0, n \in \mathbb{Z}} \hat{g}_k(n)e^{inx}\right| \leq \sum_{n\neq 0, n \in \mathbb{Z}} |\hat{g}_k(n)| \leq 2 \cdot (9/10)^k \sum_{n=1}^{\infty} |n|^{-k}.$$ 

If $k = 100$, then it follows easily that this last expression is at most $1/2$. Therefore, we get

$$|g_{100}(x) - 1| \leq 1/2.$$ 

Since $g_{100}$ is real, we see that $g_{100}(x) > 0$ for all $x$. 


6. Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a Schwartz function with $\int_{\mathbb{R}} |f|^2 = 1$ and with $\text{supp} \hat{f} \subset [1, 2]$. Suppose that $u(x, t)$ solves the Schrodinger equation $\partial_t u = i \partial^2_x u$ with initial conditions $u(x, 0) = f(x)$, (and that $u$ is Schwartz uniform).

Suppose in addition that

$$|f(x)| \leq 100(1 + |x|)^{-3}. \tag{1}$$

Prove that for all $t > 0$,

$$|u(0, t)| < 10^{-100} t^{-1}. \tag{*}$$

Physical interpretation: The solution to the Schrodinger equation models a quantum mechanical particle. The integral $\int_a^b |u(x, t)|^2 dx$ gives the probability that the particle lies in the interval $[a, b]$ at time $t$. Therefore, $|u(x, t)|^2$ can be interpreted as the ‘probability density’ that the particle is at the point $x$ at time $t$. The condition that $\text{supp} \hat{f} \subset [1, 2]$ says that the particle has “momentum” between 1 and 2. Equation 1 implies that at time 0, the particle lies fairly close to 0 with high probability. Since the particle has momentum in the range $[1, 2]$, physical intuition suggests that it is unlikely to be near zero when $t$ is large.

Solution to 6: We consider the Fourier transform of $u$. As we learned in class, $\hat{u}(\omega, t)$ obeys the differential equation

$$\partial_t \hat{u}(\omega, t) = -4\pi^2 i \omega^2 \hat{u}(\omega, t).$$

We have the initial condition $\hat{u}(\omega, 0) = \hat{f}(\omega)$, and therefore

$$\hat{u}(\omega, t) = e^{-4\pi^2 \omega^2 t} \hat{f}(\omega).$$

By Fourier inversion, we see that

$$u(0, t) = \int_{\mathbb{R}} e^{-4\pi^2 \omega^2 t} \hat{f}(\omega) d\omega.$$  

Next we want to use the hypotheses about $f$ to control $\hat{f}$. First of all, since $\hat{f}$ is supported in $[1, 2]$, we can write

$$u(0, t) = \int_1^2 e^{-4\pi^2 \omega^2 t} \hat{f}(\omega) d\omega. \tag{1}$$

The bound $|f(x)| \leq 100(1 + |x|)^{-3}$ allows us to bound both $\hat{f}(\omega)$ and its derivative. First we bound $|\hat{f}(\omega)|$. 


\[ |\hat{f}(\omega)| \leq \int_{\mathbb{R}} |f(x)| dx \leq 100 \int_{\mathbb{R}} (1 + |x|)^{-3} dx \leq 1000. \]

To bound the derivative of \( \hat{f}(\omega) \), we first recall that

\[ \frac{d}{d\omega} \hat{f}(\omega) = \int_{\mathbb{R}} (-2\pi ix)f(x)e^{-2\pi i\omega x} dx. \]

Using the estimate \( |f(x)| \leq 100(1 + |x|)^{-3} \), we get the bound

\[ \left| \frac{d}{d\omega} \hat{f}(\omega) \right| \leq 100 \int_{\mathbb{R}} (2\pi |x|)(1 + |x|)^{-3} dx \leq 10^4. \]

Now we use these estimates for \( \hat{f} \) and its derivative to control the integral in (1). We want to prove that the oscillation in the factor \( e^{-4\pi^2 \omega^2 t} \), together with the regularity of \( \hat{f} \), leads to cancellation in the integral. Because the oscillatory term has the form \( e^{-4\pi^2 \omega^2 t} \) we change variables to \( \eta = \omega^2 \). We have \( d\eta = 2\omega d\omega \), and so \( d\omega = (1/2)\eta^{-1/2} d\eta \). Therefore, the integral (1) becomes

\[ u(0, t) = \frac{1}{2} \int_{1}^{4} e^{-4\pi^2 \eta^{1/2} t} \hat{f}(\eta^{1/2})\eta^{-1/2} d\eta. \]

We abbreviate \( g(\eta) = \hat{f}(\eta^{1/2})\eta^{-1/2} \). Then we get

\[ u(0, t) = \frac{1}{2} \int_{1}^{4} g(\eta)e^{-4\pi^2 \eta^{1/2} t} d\eta. \]

We will control this integral by integrating by parts. We integrate by parts with \( u = g(\eta) \) and \( dv = e^{-4\pi^2 \eta^{1/2} t} d\eta \). We note that \( g \) is a \( C^1 \) function on \([1, 4]\), and that \( g \) vanishes at the endpoints of \([1, 4]\). Because of this vanishing, the boundary terms vanish when we integrate by parts, and we get

\[ u(0, t) = \frac{1}{2} \int_{1}^{4} g(\eta)e^{-4\pi^2 \eta^{1/2} t} d\eta = -\frac{1}{-8\pi^2 it} \int_{1}^{4} g'(\eta)e^{-4\pi^2 \eta^{1/2} t} d\eta. \]

Therefore, we get

\[ |u(0, t)| \leq t^{-1} \max_{\eta \in [1,4]} |g'(\eta)|. \]

It just remains to bound \( |g'(\eta)| \). Using the Liebniz rule and the chain rule, we see that

\[ g'(\eta) = \left( \hat{f}(\eta^{1/2})\eta^{-1/2} \right)' = \hat{f}'(\eta^{1/2}) \cdot \left( \frac{1}{2} \eta^{-1/2} \right) \eta^{-1/2} + \hat{f}(\eta^{1/2}) \left( \frac{-1}{2} \eta^{-3/2} \right), \]
When $\eta \in [1, 4]$, negative powers of $\eta$ are at most 1. Using our bounds $|\hat{f}(\omega)| \leq 10^3$ and $|\hat{f}'(\omega)| \leq 10^4$, we see that

$$\max_{\eta \in [1, 4]} |g'(\eta)| \leq 10^4 + 10^3 \leq 10^5.$$ 

Therefore, we get all together

$$|u(0, t)| \leq 10^5t^{-1}.$$ 

Final remarks: If $f$ decays faster, we can prove even better decay for $|u(0, t)|$. Given a bound of the form

$$|f(x)| \leq C(1 + |x|)^{-m},$$

we can bound $|\hat{f}(\omega)|$ and we can bound the derivatives $|\frac{d^k}{d\omega^k} \hat{f}(\omega)|$ for $1 \leq k \leq m - 2$. Following the same strategy and integrating by parts $m - 2$ times, we can prove the following stronger bound for $|u(0, t)|$:

$$|u(0, t)| \leq C't^{-(m-2)}.$$