1. Strauss 12.4, p. 352: # 1, 2.

2. For the heat equation \( u_t = \Delta u \) on \( \mathbb{R}^n \), use Plancherel’s formula to prove that
\[
\int_{\mathbb{R}^n} |u(x, t)|^2 dV(x) \leq \int_{\mathbb{R}^n} |u(x, 0)|^2 dV(x).
\]
What is the limit of the left-hand side as \( t \to \infty \)?

3. Suppose \( u(x) \) is a harmonic function on \( \mathbb{R}^n \) of polynomial growth, i.e., \( |u(x)| \leq C(1 + |x|^N) \), for some constant \( C \) and \( N \in \mathbb{N} \). With the help of the Fourier transform, show that \( u(x) \) is a polynomial. You may use the following fact.

**Theorem.** If \( f \in \mathcal{D}'(\mathbb{R}^n) \) has \( \text{supp } f \subset \{0\} \), then there exists a number \( k \in \mathbb{N} \) and constants \( \{c_\alpha\}_{|\alpha| \leq k} \) so that
\[
f = \sum_{|\alpha| \leq k} c_\alpha \partial^\alpha \delta.
\]

*Hint: The assumptions (which ones?) imply that \( u \) (hence also \( \hat{u} \)) may be regarded as an element of \( \mathcal{S}' \subset \mathcal{D}' \). What is the support of \( \hat{u} \)?*

4. Solve
\[
\left( \frac{\partial^2}{\partial t^2} + \Delta \right) u(x, t) = 0
\]
for \( t \geq 0 \), given \( u(x, 0) = f(x), u_t(x, 0) = g(x) \), for \( f, g \in \mathcal{S} \) with \( |u(x, t)| \leq C(1 + |t|) \). *Hint: Show*
\[
(\mathcal{F}_x u)(\xi, t) = te^{-t|\xi|} \hat{g}(\xi) + \left( e^{-t|\xi|} + t|\xi|e^{-t|\xi|} \right) \hat{f}(\xi).
\]
To invert, note that
\[
\mathcal{F}_x^{-1}(|\xi|e^{-t|\xi|}) = -\frac{\partial}{\partial t} \mathcal{F}_x^{-1}(e^{-t|\xi|}).
\]

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