18.152 – Problem Set 5

1. For any \( \mathbf{v} \in \mathbb{R}^n \), the translation operator, \( \tau_{\mathbf{v}} \), acts on functions \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) by \((\tau_{\mathbf{v}}f)(\mathbf{x}) = f(\mathbf{x} + \mathbf{v})\).
   
   (a) Considering first its action on \( L^1_{\text{loc}}(\mathbb{R}^n) \), define an extension \( \tau_{\mathbf{v}} : \mathcal{D}' \rightarrow \mathcal{D}' \) and prove that your operator is well-defined.
   
   (b) Let \( f \in \mathcal{D}' \). Using \( \tau_{\mathbf{v}} \) from (a), what is \( \lim_{h \to 0} \left( \frac{1}{h} \left( (\tau_{h\mathbf{v}}f) - f \right) \right) \)?
   
   (c) Use (b) to recompute \( \lim_{n \to \infty} f_n \) from \# 2 of Problem Set 4.

2. Let \( a, b \) be non-negative numbers. Compute \( e^{-a|x|^2} * e^{-b|x|^2} \) by means of the Fourier transform.

3. Suppose \( f \in \mathcal{S} \) and \( p \) is a polynomial. Show \( f * p \) is a polynomial.

4. Suppose \( f \in L^1 \) is a radial function, i.e., \( f(A\mathbf{x}) = f(\mathbf{x}) \), for all orthogonal \( A \). Show that \( \hat{f} \) is also a radial function.

5. Let \( \mathcal{E} \) denote the class of piecewise continuous functions on \( \mathbb{R} \) that vanish for \( t < 0 \) and satisfy a growth estimate of the form \( |f(t)| \leq Ce^{at} \) for some \( a \in \mathbb{R} \).
   
   For a function \( f \in \mathcal{E} \), the Laplace transform of \( f \) is defined to be
   
   \[ \mathcal{L}[f](z) := \int_0^\infty f(t)e^{-zt} dt, \]
   
   for \( z \in \mathbb{C} \) with \( \Re z > a \).
   
   (a) If \( f \) is continuous and piecewise smooth, with \( f' \in \mathcal{E} \), prove
   
   \[ \mathcal{L}[f'](z) = z\mathcal{L}[f](z) - f(0). \]
   
   (b) Prove
   
   \[ \mathcal{L}[tf(t)](z) = -(\mathcal{L}[f])'(z). \]