18.152 – Problem Set 3


2. Strauss 2.4, p. 53: # 9. From your answer, deduce the value of the integral
\[ \int_{-\infty}^{\infty} s^2 e^{-s^2} \, ds. \]

3. Show that if \( u_t - u_{xx} = 0 \) on \([-1, 1]\) with \( u(\pm 1, t) = e^{2t} \) and \( u(x, 0) \geq 1 \), then \( u(x, t) \geq x^2 + 2t \) for all \( x \in [-1, 1] \) and \( t > 0 \).

4. Show that
\[ u(x, t) = \sum_{k=0}^{\infty} f^{(k)}(t) \frac{x^{2k}}{(2k)!}, \]
where
\[ f(t) = \begin{cases} e^{-t^2} & \text{if } t \neq 0 \\ 0 & \text{otherwise,} \end{cases} \]
is a \( C^\infty \)-function satisfying the heat equation \( u_t = u_{xx} \) on \( \mathbb{R} \times \mathbb{R} \) with \( u(x, 0) \equiv 0 \), but \( u(0, t) > 0 \) for \( t \neq 0 \).

5. Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set with \( C^1 \)-boundary \( \partial \Omega \) and \( u \in C^2(\overline{\Omega} \times (0, T)) \cap C(\overline{\Omega} \times [0, T]) \) a solution to \( u_t = \Delta u \) in \( \Omega \times (0, T) \) with homogeneous Dirichlet conditions.

   (a) Define
   \[ E(t) = \int_{\Omega} u^2(x, t) \, dV(x), \]
   and use Hölder’s inequality to prove that \( E''(t)E(t) \geq (E'(t))^2 \).

   (b) Prove that, where \( E(t) > 0 \), the function \( l(t) \equiv \log E(t) \) is convex in \( t \).

   (c) Conclude that if \( u(x, T) \equiv 0 \), then \( u(x, t) \equiv 0 \) on all of \( \Omega \times [0, T] \).

   (d) From this, formulate and prove a “backwards-uniqueness” property for the Dirichlet problem for the inhomogeneous heat equation on \( \Omega \).