Solutions for Quiz #1, 03/9/07.

Problem Q1.1 (25 pts). Solution. The unsolvable system is \( A\hat{u} = b \), where

\[
A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \hat{u} = \begin{bmatrix} C \\ D \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.
\]

Form the normal equation,

\[
A^T A \hat{u} = A^T b: \quad \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \]

which is solved by \( C = -2/5 \) and \( D = 4/5 \), so that the best fit is \( y = -2/5 + 4/5x \).

Problem Q1.2 (25 pts). Solution. 1. Subtract twice row 2 from row 3 and row 1 from row 4 to get

\[
U = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

We see three pivots equal to 1 and a 0 in the fourth pivot position. The matrix is positive semi-definite. 2. To find \( x^T Ax \) we write \( A \) as \( LDL^T \) and then

\[
x^T Ax = (L^T x)^T D (L^T x) = (x_1 + x_4)^2 + (x_2 + 2x_3)^2 + x_3^2.
\]

3. Pick any three of the following four: i) A symmetric matrix can have only real eigenvalues; ii) The product of the eigenvalues must be zero because of the zero in
the pivot position; iii) The sum of the eigenvalues must equal \( \text{Tr} (\mathbf{A}) = 8 \); iv) Positive semidefinite matrix cannot have negative eigenvalues.

**Problem Q1.3 (25 pts). Solution.** 1. The Green’s function is linear at 
\(-1 < x < x_0\) and at \( x_0 < x < 1 \), is continuous at \( x_0 \) and has drop in the slope by 1 across \( x_0 \). 
\[
G(x, x_0) = \begin{cases} 
A + Bx, & -1 \leq x \leq x_0 \\
C + Dx, & x_0 \leq x \leq 1 
\end{cases}
\]
with \( A + Bx_0 = C + Dx_0 \), \( B = D + 1 \). Application of boundary conditions gives: \( A - B = 0 \) and \( C + D = 0 \).

From here we find \( A = B = \frac{1-x_0}{2} \), \( C = -D = \frac{1+x_0}{2} \) and the final Green’s function is

\[
G(x, x_0) = \begin{cases} 
\frac{1}{2} (1-x_0) (1+x), & -1 \leq x \leq x_0 \\
\frac{1}{2} (1+x_0) (1-x), & x_0 \leq x \leq 1 
\end{cases}
\]

2. \( \mathbf{A}_4 \) is simply \( \mathbf{K}_4 \) and

\( h = 2/(n+1) = 2/5 \). 3. The columns of \( \mathbf{A}_4^{-1} \) are the samples of the Green’s function at the grid points.

**Problem Q1.4 (25 pts). Solution.** One of three possible ways (see solutions to Pset #1) to solve the problem is by testing the formula for \( u = 1, x, x^2 \) for which the formula must be exact. Then we test with \( u = 1, u = ih \), and \( u = (ih)^2 \) at \( i = 0 \):

\[
0 = \frac{1}{h} \left( -\frac{3}{2} \cdot 1 + 2 \cdot 1 \cdot h - \frac{1}{2} \cdot 1 \cdot h \right) = 0, 
1 = \frac{1}{h} \left( -\frac{3}{2} \cdot 0 + 2h - \frac{1}{2} \cdot 2h \right) = 1, 
0 = \frac{1}{h} \left( -\frac{3}{2} \cdot 0 + 2 \cdot h^2 - \frac{1}{2} \cdot (2h)^2 \right) = 0.
\]

To find \( b \) (for \( u = x^3 \)) and \( n \), plug in 

\[
u = x^3 = (ih)^3: 0 = \frac{1}{h} \left( -\frac{3}{2} \cdot 0 + 2 \cdot h^3 - \frac{1}{2} \cdot (2h)^3 \right) + bh^n.
\]

Then \(-2h^2 + bh^n = 0\), from which we find \( b = 2 \) and \( n = 2 \). The above formula is a second order approximation of \( du/dx \). More generally, use Taylor series:

\[
u_{i+1} = u(x+h) = u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \frac{1}{6}u'''(x)h^3 \quad \text{and similarly for} \quad u_{i+2} = u(x+2h) \text{ retaining terms of order} \ h^3.
\]

Then form the combination

\[-\frac{3}{2}u(x) + 2u(x+h) - \frac{1}{2}u(x+2h) \text{ to find out that} \ n = 2 \text{ and} \ b = \frac{1}{3}u'''(x).\]