Section 1.1 - Ex. 1

Let \( B \) be the collection of all Cauchy sequences. Show that \( B \) is uncountable.

Proof: Assume that \( B \) is countable.

Then there exists a bijection \( f : \mathbb{N} \to B \) that attaches to every element in \( B \) an index in \( \mathbb{N} \). (A)

Now, let \( \omega = \omega_1 \omega_2 \cdots \) be an element of \( B \) such that \( \omega_i = f(i) \), \( \forall i \in \mathbb{N} \)

Clearly such a \( \omega \) exists in \( B \),

namely \( f^{-1}(\omega) = i \Rightarrow \omega_i = f(i) \), for any \( i' \) such \( \omega_i \neq f(i') \), then \( f^{-1}(\omega) \) is not defined.

Thus \( f \) is not onto and so \( B \) is not a bijection.

Therefore such an \( f \) cannot exist

i.e. (A) fails

\( \therefore B \) is not countable, i.e. \( B \) is uncountable.
Section 1.1

To compute $S_n(w)$:

$$S_n(w) = \sum_{i=1}^{n} R_i(w)$$

Now, we started with $-1$, so you'll be ruined at time $k$ if

$$S_k(w) = -1$$

and you've never been ruined "before" that, i.e., $S_i(w) > -1$ for $k > i$.

So,

$$S_0(w) = \begin{array}{c}
\mid \hspace{1cm} \mid \hspace{1cm} \mid \\
0 & \text{ out of 2} \\
\end{array}$$

$$S_1(w) = \begin{array}{c}
\mid \hspace{1cm} \mid \hspace{1cm} \mid \\
0 & \text{ out of 2} \\
\mid \hspace{1cm} \mid \hspace{1cm} \mid \\
-1 & \text{ out of 6} \\
\end{array}$$

$$S_2(w) = \begin{array}{c}
\mid \hspace{1cm} \mid \hspace{1cm} \mid \\
0 & \text{ out of 2} \\
\mid \hspace{1cm} \mid \hspace{1cm} \mid \\
1 & \text{ out of 8} \\
\end{array}$$

$$S_3(w) = \begin{array}{c}
\mid \hspace{1cm} \mid \hspace{1cm} \mid \\
0 & \text{ out of 2} \\
\mid \hspace{1cm} \mid \hspace{1cm} \mid \\
2 & \text{ out of 12} \\
\mid \hspace{1cm} \mid \hspace{1cm} \mid \\
3 & \text{ out of 32} \\
\end{array}$$

So at time 1, prob of ruin = $\frac{1}{2} = 0.5$ ($= M\left(\frac{1}{2}, \frac{1}{2}\right)$)

at time 3, prob of ruin = $\frac{1}{8} = 0.125$ ($= M\left(\frac{1}{2}, \frac{1}{2}\right)$)

at time 5, prob of ruin = $\frac{1}{32} = 0.0625$ ...

Now, $\text{Prob}(\text{Expected Ruin}) = \text{Prob}\left(\bigcup_{k=1}^{\infty} \text{Ruin at Time } k\right)$

at "Ruin at Time $k$" is an disjoint event.

$$= \sum_{k=1}^{\infty} \text{Prob}(\text{Ruin at Time } k)$$
\[ \geq \text{Prob (win at 1)} + \text{Prob (win at 3)} + \text{Prob (win at 5)} + \text{Prob (win at 7)} \]

We have 1, 3 and 5.

Regarding 7:
- At \( S_5(w) \) we had five "1" intervals, that was of interest.
- So going to \( S_7(w) \) we'll get five "0" intervals.
- And going to \( S_9(w) \) yields five "1" intervals.
  
  \[ \text{Out of } 2^7 = 128 \]

\[ \text{Prob (win at 7)} = \frac{5}{128} = 0.0390625 \geq 0.039 \]

So \( \text{Prob (eventual Rain)} \geq 0.5 + 0.125 + 0.0625 + 0.039 \]

\[ = 0.7265 \]

\[ \Rightarrow 70\% \]

\[ \text{Prob (eventual Rain)} > 70\% \]
Show that \[ \int R_1 R_2 \ldots R_n \, dx = 0 \quad \text{or} \quad 1 \]

\[ \text{or} \quad R_1 \leq R_2 \leq \ldots \leq R_n \]

"Quick idea in English."

All \( R_i \)s oscillate having frequencies which are multiple of each other. In other words, take the \( R_i \)s having the largest index; all \( R_j \)s having a \( j < i \) are constant during intervals where \( R_i \) oscillates having \( i \) as such as \( 1 \).

i.e. when we want to integrate, the whole support consists of a union of such intervals and integrating over every interval yields a zero. This holds while you have an odd number of \( R \) having this largest index.

If you have an even number of them, their product is simply 1. So redo the procedure with the next largest index.

Correlating the problem: given \( \{ \delta_i \} \), such that \( \delta_j \leq \delta_k \) for \( j < k \)

Let \( \Gamma_n = \left\{ i : i \in \mathbb{N} \text{ and } i \leq n, 1, \ldots, n \right\} \) be the collection of indices.

Define \( I(\Gamma_n) = \int \prod_{i \in \Gamma_n} R_i \, dx = \int R_1 R_2 \ldots R_n \, dx \).
Now, define \( \Gamma_n = \{ e \in \Gamma_n' : Re = \gamma_n \} \).

This is simply the indices leaving the same value.

Then
\[
I(\Gamma_n') = \begin{cases} \overline{I(\Gamma_n' - \Gamma_n)} & \text{if } |\Gamma_n'| \text{ is even} \\ 0 & \text{if } |\Gamma_n'| \text{ is odd} \end{cases}
\]

For cases:
\[
I(\Gamma_0') = 1 \quad \text{(To would correspond To } \emptyset \text{)}
\]

we can only get to "To" if only get a succession of even indices.

i.e., they all multiply but to 1!!

If we happen to go to something odd, we'll get a zero as a result.
we have $R_k(n+1) = R_k(a)$

$$R_k(a) = \begin{cases} 1 & \text{if } a \in \bigcup_{i=0}^{\infty} \left[ \frac{2i+1}{2^{k+1}}, \frac{2i+2}{2^{k+1}} \right] \\ -1 & \text{if } a \in \bigcup_{i=0}^{\infty} \left[ \frac{2i}{2^{k+1}}, \frac{2i+1}{2^{k+1}} \right] \end{cases}$$

Claim

$$R_{k-1}(a) = \begin{cases} 1 & \text{if } a \in \bigcup_{i=0}^{\infty} \left[ \frac{2i+1}{2^{k+1}}, \frac{2i+2}{2^{k+1}} \right] \\ -1 & \text{if } a \in \bigcup_{i=0}^{\infty} \left[ \frac{2i}{2^{k+1}}, \frac{2i+1}{2^{k+1}} \right] \end{cases}$$

$$R_{k-1}(a) = R_k(2a) \quad \square$$

Show $R_k(a) = R_k(2^{k-1}a)$:

we have $R_{k-1}(a) = R_k(2a)$ (Claim)

Therefore by induction,

$$R_k(a) = R_{k-1}(2a)$$
$$= R_{k-2}(2a)$$
$$= \ldots$$
$$= R_1(2^{k-1}a)$$

so $R_k(a) = R_1(2^{k-1}a)$
Want to show that

\[ R_n(x) = -\sqrt{n} \sin(2\pi x - \pi) \quad \text{except at a finite number of points} \]

for \( 0 < x < 1 \):

\[ -\sqrt{n} \sin(2\pi x - \pi) = -1 \iff \sin(2\pi x - \pi) > 0 \]

\[ \iff 2k\pi < 2\pi x - \pi < \pi + 2k\pi \quad \forall k \in \mathbb{Z} \]

\[ \iff \frac{k}{2} < x < \frac{k + 1}{2} \quad \forall k \in \mathbb{Z} \text{ st. } 0 < k < 1 \]

\[ \iff R_n(x) = -1 \quad \text{and} \quad x \neq \frac{k}{2} \quad \forall k \in \mathbb{Z} \text{ st. } 0 < k < 1 \]

(\( k = 1 \) is not included!)

Note that

\[ -\sqrt{n} \sin(\pi x - \pi) = 1 \iff -\sqrt{n} \sin(\pi x) = 1 \]

\[ \iff R_n(x) = 1 \quad \forall k \in \mathbb{Z} \text{ st. } 0 < x < 1 \]

\[ \forall k \in \mathbb{Z} \text{ st. } 0 < x < 1 \]
Therefore,

\[ R_n(x) = -\text{sgn} \left[ \sin \left( 2\pi 2^{n-1} x \right) \right] \]

except at \( 2^{n-1} \) points that correspond to

\[ 0, \frac{1}{2^{n-1}}, \frac{2}{2^{n-1}}, \frac{3}{2^{n-1}}, \ldots \]
Section 1.1: Ck id.

Let C be the Cantor set.

Show that C is uncountable.

Proof: Assume C is countable.

Then there exists a bijection \( f : \mathbb{N} \rightarrow C \) that assigns to each element in \( C \) an index in \( \mathbb{N} \).

Now, let \( w = (a_0, a_1, a_2, \ldots) \) be an element of \( C \)

such that \( a_k \neq f(k) \) for some \( k \in \mathbb{N} \).

Clearly such a \( w \) exists in \( C \).

"Simply go along the diagonal and offset 2 to 0, 0 to 2" because \( f(w) = \) a 1 = a(f(1)) for any i, but a i \neq f(i) at least once, \( f(w) \) is not defined, and \( f \) is not a bijection.

Thus, such an \( f \) cannot exist.

i.e. (C) fails.

\[ \therefore \ C \ \text{is not countable, i.e. } C \ \text{is uncountable}. \]
Section 1.3

X uncountable set
R = collection of all finite subsets of X
\( m(A) \) of \( A \in R \), the number of elements in A.

Show that R is a ring

Let \( A, B \in R \) be two sets.

- then \( |A| < \infty \) and \( |B| < \infty \)

1) \( A \cup B \in R \)
   \( \Rightarrow \): \( |A \cup B| \leq |A| + |B| < \infty \)

2) \( A - B \in R \)
   \( \Rightarrow \): \( |A - B| \leq |A| < \infty \) (since \( A \cap B \subseteq A \))

Since 1) and 2) then \( R \) is a ring.

Show that \( \mu \) is a measure on \( R \)

We simply need to show that \( \mu \) is a countably additive, non-negative set function.

a) \( \mu \) is non-negative.
   \( \Rightarrow \): Let \( A \in R \) be a set, \( \mu(A) = |A| \geq 0 \)

b) \( \mu \) is countably additive, i.e.
   Given any countable collection \( \{A_i\}_{i=1}^{\infty} \subset R \) of \( A_i \)'s mutually disjoint and such that \( A = \bigcup_{i=1}^{\infty} A_i \) is also in \( R \), then \( \mu(A) = \sum_{i=1}^{\infty} \mu(A_i) \)
Lemma 1: \( M \) is finitely additive.

If \( A_1, A_2 \in \mathbb{R} \) are (mutually) disjoint,

then \( M(A_1 \cup A_2) = |A_1 \cup A_2| = |A_1| + |A_2| = M(A_1) + M(A_2) \)

By induction, we can extend this property as follows:

If \( A_1, A_2, \ldots, A_n \in \mathbb{R} \) are mutually disjoint, then

\[
M(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} M(A_i)
\]

Consider a set \( A \in \mathbb{R} \).

Now, let \( \{A_i\}_{i=1}^{\infty} \) be a disjoint collection of sets in \( \mathbb{R} \) such that \( A = \bigcup_{i=1}^{\infty} A_i \).

Then

(i) \( M(A) \geq \sum_{i=1}^{\infty} M(A_i) \)

\[ \text{Proof: } \bigcup_{i=1}^{N} A_i \subseteq \bigcup_{i=1}^{\infty} A_i = A \quad \text{for every } N \]

So \( M(A) = |A| \geq |\bigcup_{i=1}^{N} A_i| = \mu\left(\bigcup_{i=1}^{N} A_i\right) \quad \text{for every } N \)

Then,

\[
M(A) \geq \sum_{i=1}^{\infty} M(A_i)
\]

(ii) \( M(A) \leq \sum_{i=1}^{\infty} M(A_i) \)

\[ \text{Proof: } A \in \mathbb{R}, \text{ then } |A| < \infty \]

i.e. \( A \) is finite collection of elements in \( X \)

Then \( \exists C \in \mathbb{N} \) such that \( A = \bigcup_{i=1}^{C} A_i \) and \( |\delta| < \frac{a}{6} \)

In other words, we do not need the whole \( \{A_i\} \) sequence.

To construct \( A' \):

all but finite are empty.
(Small proof: \( \{ A_i \} \) is a disjoint collection of sets in \( \mathbb{R} \).)

Going from \( \bigcup A_i \) to \( A_i \), we:

Never add new elements
2) add nothing

1) may occur infinitely many times since \(|A| < \infty\)
2) when 2) occurs, this means that the added \( A_i \) is the empty set \( \emptyset \), therefore \( A_i \)'s are not disjoint.

So we can only keep those used in 1)
and so we get a finite collection)

Then \( \mu(A) = \mu(\bigcup A_i) = \sum_{i \in S} \mu(A_i) \leq \sum_{i=1}^{\infty} \mu(b_i) \)

Since 1) and 2) we get:

\[ \mu\left( \bigcup_{i \in \mathbb{I}} A_i \right) = \sum_{i \in \mathbb{I}} \mu(A_i) \]

Since a) and b) then \( \mu \) is a measure.

Identity \( \mu^* \)

If \( A \in \mathbb{R} \), then \( \mu^*(A) = \mu(A) \)

If \( A \notin \mathbb{R} \), then \( A \) is either 1) countably negligible
in this case, by countable additivity we get \( \mu^*(A) = +\infty \)
2) uncountable
in this case \( \mathbb{L} \) is an outer measure \( \mu(A) = \emptyset \)
we get \( \mu^*(A) = +\infty \)
What are $\mathcal{M}$ and $\mathcal{M}_R$?

$\mathcal{M}$ is the closure of $\mathcal{R}$ in $\mathcal{M}_R$. In this case, $\mathcal{M}$ will be the collection of all countable sets in $\mathcal{R}$. $\mathcal{M}_R \subseteq \mathcal{R}$ finite (since $\mathcal{R}$ has to be finite).

And $\mathcal{M}$ would be the collection of all countable (finite or infinite) sets in $\mathcal{M}_R$.

Is every subset of $\mathcal{M}_R$ measurable?

No, only countable sets in $\mathcal{M}_R$ are measurable.
Section 1.3

$\mathbb{R} = \mathbb{Q}$

If $A \subseteq \mathbb{R}$, then $\mu(A) = 1$ if for some $\varepsilon > 0$ $A$ contains $(0, \varepsilon)$

$\mu(A) = 0$ otherwise.

Show that $\mu$ is an additive set function, but is not countably additive.

$\mu$ is additive: If $A_1, A_2, \ldots, A_n \subseteq \mathbb{R}$ are mutually disjoint

then at most one of the $A_i$'s will contain $(0, \varepsilon)$ for some $\varepsilon > 0$

Case 1: $\bigcup_{i=1}^n A_i \supseteq (0, \varepsilon)$ for some $\varepsilon > 0$

Then $\mu\left(\bigcup_{i=1}^n A_i\right) = 1$ and $\sum_{i=1}^n \mu(A_i) = \mu(A_k)$ where $A_k \supseteq (0, \varepsilon)$ for some $\varepsilon > 0$

Case 2: $\bigcup_{i=1}^n A_i \nsubseteq (0, \varepsilon)$ for all $\varepsilon > 0$

$\mu\left(\bigcup_{i=1}^n A_i\right) = 0$ and $\sum_{i=1}^n \mu(A_i) = \sum_{i=1}^n 0 = 0$

The cases are exhaustive. Then $\mu\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i)$

$\mu$ is not countably additive: Consider $A = (0, 1) \subseteq \mathbb{R}$

Take $A_i = \left(\frac{i}{2^n}, \frac{i+1}{2^n}\right)_{n=1}^\infty$

Then $\bigcup_{i=1}^\infty A_i \subseteq \mathbb{R}$ is a countable collection with the $A_i$'s

mutually disjoint and such that $A = \bigcup_{i=1}^\infty A_i$

Now, $\mu(A) = \mu((0, 1)) = 1$

but $\sum_{i=1}^\infty \mu(A_i) = \sum_{i=1}^\infty 0 = 0$

$\int \infty$ countably additivity does not always hold.
Sect. 1.3

Our \( \mu \) is continuous, monotone increasing. 

\[ \mu_A(A) = \sum_{i=1}^{n} \mu_i(A_i) \]

Define that \( \mu_A \) is countably additive: 

\[ \mu_A(A) = \sum_{i=1}^{\infty} \mu_i(A_i) \]

The proof here will follow the same lines as that of the book.

If \( A = \bigcup_{i=1}^{n} A_i \) is a collection of disjoint elements \( \text{wt. } A \), \( \Rightarrow \)

\[ \mu_A(A) = \sum_{i=1}^{n} \mu(A_i) \]

If \( A = \bigcup_{i=1}^{n} A_i \) is a collection of disjoint elements \( \text{wt. } A \), \( \Rightarrow \)

\[ \mu_A(A) = \sum_{i=1}^{n} \mu(A_i) \]

Lemma: Let \( A \in \mathcal{R} \) and let \( \epsilon > 0 \) be given.

There exists \( F, G \in \mathcal{R} \) such that \( F \) is closed, \( G \) is open,

\[ F \subseteq A \subseteq G, \text{ and } \mu(G) \leq \mu(F) + \epsilon \]

\[ \mu(F) \leq \mu(G) \leq \mu(F) + \epsilon \]

\[ \mu(F) = \mu(G) \leq \mu(F) + \epsilon \]
Proof of Lemma: Suppose \( A \) is a single interval \([a, b]\)

Then \( \forall \varepsilon > 0, \exists \delta > 0 \)

**Case I:**

\[
F(b - \delta) - F(a + \delta) \geq F(b) - F(a) - \varepsilon
\]

\[
F(c + \delta) - F(a - \delta) \leq F(b) - F(a) + \varepsilon
\]

Thus, we get three equations, three unknowns.

(please do note that the constraints for \( \varepsilon \) and \( \delta \)

are not binding, so a feasible point exists)

**Case II:** One may argue identically as in Case I.

Then let \( F = [a + \delta, b - \delta] \)

\[
G = (a - \delta, b + \delta)
\]

we have,

\[
F_{i}(G) \geq F_{i}(A) - \varepsilon
\]

\[
F_{i}(G) \leq F_{i}(A) + \varepsilon
\]

Now, suppose \( \bigcup_{i=1}^{k} A_{i} \) is a disjoint union of

intervals, then \( F_{i}, G_{i} \) for each \( A_{i}, \forall i \)

\[
\forall A_{i}, \text{ closed}, G_{i}, \text{ open}
\]

and \( F_{i}(G_{i}) \geq F_{i}(A_{i}) - \frac{\varepsilon}{k} \)

\[
F_{i}(G_{i}) \leq F_{i}(A_{i}) + \frac{\varepsilon}{k}
\]

then let \( F = \bigcup_{i=1}^{k} F_{i} \) and \( G = \bigcup_{i=1}^{k} G_{i} \), we'll have
\[ \mu_F(F) = \sum_{i=1}^{k} \mu_F(A_i) \geq \sum_{i=1}^{k} \left[ \mu_F(A_i) - \frac{\varepsilon}{k} \right] = \mu_F(A) - \varepsilon \]

\[ \mu_F(A) \leq \sum_{i=1}^{k} \mu_F(A_i) \leq \sum_{i=1}^{k} \left[ \mu_F(A_i) + \frac{\varepsilon}{k} \right] = \mu(A) + \varepsilon \]

so \[ \mu_F(F) \geq \mu_F(A) - \varepsilon \]

\[ \mu_F(A) \leq \mu_F(A) + \varepsilon \]

Notice, for the given \( A \), pick a closed set \( F \subseteq A \)

So, \[ \mu_F(F) \geq \mu_F(A) - \frac{\varepsilon}{2} \]

and for each \( A_i \), choose an open set \( G_i \) containing \( A_i \) with \[ \mu(G_i) \leq \mu(A_i) + \frac{\varepsilon}{2^i+1} \]

\( F \) is closed and bounded, then \( F \) is compact, so any open cover of \( F \) has a finite subcover.

But \( F \subseteq A \subseteq \bigcup_{i=1}^{N} G_i \), then \( \{G_i\}_{i=1}^{N} \) is an open cover of \( F \), and so we only need a finite union of \( G_i \)'s. No cover \( F \) namely \( \{G_i\}_{i=1}^{N} \)

So \[ \mu_F(A) - \frac{\varepsilon}{2} \leq \mu_F(F) \leq \mu_F\left( \bigcup_{i=1}^{N} G_i \right) \leq \sum_{i=1}^{N} \mu_F(G_i) \]

\[ \leq \sum_{i=1}^{N} \left[ \mu_F(A_i) + \frac{\varepsilon}{2^i+1} \right] \]

\[ \leq \sum_{i=1}^{N} \mu_F(A_i) + \frac{\varepsilon}{2} \]

\[ \therefore \mu_F(A) \leq \sum_{i=1}^{N} \mu_F(A_i) + \varepsilon \]

So, \( \mu_F(A) \leq \sum_{i=1}^{N} \mu_F(A_i) + \varepsilon \n\]

Since \( \varepsilon \) is fixed \( \forall \varepsilon > 0 \), \[ \mu_F(A) \leq \sum_{i=1}^{\infty} \mu_F(A_i) \]

\[ \square \] and \( \square \Rightarrow \mu_F(A) = \sum_{i=1}^{\infty} \mu_F(A_i) \) i.e. \( \mu_F \) is countably additive.