The Stability of the Irrotational Euler-Einstein System with a Positive Cosmological Constant

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Indices

Greek indices $\in \{0, 1, 2, 3\}$
Latin indices $\in \{1, 2, 3\}$

\[ \partial_t = \partial_0 \]
\[ \partial = (\partial_t, \partial_1, \partial_2, \partial_3) \]
\[ \bar{\partial} = (\partial_1, \partial_2, \partial_3) \]
Einstein’s equations

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \]

- We fix \( \Lambda > 0 \)
- Bianchi identities imply \( D_\mu T^{\mu\nu} = 0 \)
- Our spacetimes will have topology \([0, T] \times \mathbb{T}^3\)
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The notion of “stability"

- Need a background solution (Ours will be FLRW type)
  - Initial value problem formulation of Einstein equations
    - We will use wave coordinates (de Donder 1921, Choquet-Bruhat, 1952)
  - Goal: show that if we slightly perturb the background data, then the resulting solution exists for all $t \geq 0$ and that the spacetime is future causally geodesically complete
  - We show convergence as $t \to \infty$
  - Our proofs are based on energy estimates for quasilinear wave equations
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Previous stability results

For $\Lambda = 0$:

- Vacuum Einstein using maximal foliation (Christodoulou & Klainerman, 1993)
- Einstein-Maxwell (Zipser, 2000)
- Vacuum Einstein using double-null foliation (Klainerman & Nicolò, 2003)
- Einstein-scalar field using wave coordinates (Lindblad & Rodnianski, 2004-2005)
- Einstein-Maxwell-scalar field in $1+n$ dimensions (Loizelet, 2008)
- Vacuum Einstein for more general data (Bieri, 2009)
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For $\Lambda = 0$ using wave coordinates

- Einstein-scalar field with a potential $V(\Phi)$ in $(1 + n)$ dimensions (Ringström, 2008)
- $\Box \Phi = V'(\Phi)$
- $V(\Phi)$ emulates $\Lambda > 0$ : $V(0) > 0$, $V'(0) = 0$, $V''(0) > 0$
Previous stability results cont.

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- Euler-Poisson with a cosmological constant (Brauer, Rendall, & Reula, 1994)
Why $\Lambda > 0$?

Answer #1: Einstein was the first to envision $\Lambda$: he was looking for static solutions.

Answer #2: In 1929, Hubble formulated an empirical “law,” which was based on observations of the redshift effect, and which suggested that the universe is expanding. Hubble’s “law:” galaxies are receding from Earth, and their velocities are proportional to their distances from it.

In the 1990’s: data from type Ia supernovae and the Cosmic Microwave Background suggested accelerated expansion.

$\Lambda > 0 \implies \exists$ solutions with accelerated expansion. e.g.

$$g = -dt^2 + e^{2(\sqrt{\Lambda/3})t} \sum_{a=1}^{3} (dx^a)^2$$
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The modified irrotational Euler-Einstein system

\[ h_{jk} \overset{\text{def}}{=} e^{-2\Omega} g_{jk}, \quad W \overset{\text{def}}{=} 3/(1 + 2s), \quad H^2 \overset{\text{def}}{=} \frac{\Lambda}{3}, \quad \hat{\Box} g \overset{\text{def}}{=} g^{\alpha\beta} \partial_{\alpha} \partial_{\beta} \]

\[ \hat{\Box} g (g_{00} + 1) = 5H \partial_t g_{00} + 6H^2 (g_{00} + 1) + \triangle_{00} \]
\[ \hat{\Box} g g_{0j} = 3H \partial_t g_{0j} + 2H^2 g_{0j} - 2H g^{ab} \Gamma_{ajb} + \triangle_{0j} \]
\[ \hat{\Box} g h_{jk} = 3H \partial_t h_{jk} + \triangle_{jk} \]
\[ -\partial_t^2 \Phi + m^{jk} \partial_j \partial_k \Phi + 2m^{0j} \partial_t \partial_j \Phi = W \omega \partial_t \Phi + \triangle(\partial \Phi) \]

\( m^{\mu\nu} \) is the reciprocal acoustic metric

The "\( \triangle \)" terms are error terms
The appearance of $z_j$ in the fluid equation

\[-\partial_t^2 \Phi + m^{jk} \partial_j \partial_k \Phi + 2m^{0j} \partial_t \partial_j \Phi = w \omega \partial_t \Phi + \triangle (\partial \Phi)\]

\[m^{jk} = \frac{g^{jk} - \triangle^{jk}_{(m)}}{(1 + 2s) + \triangle (m)}\]

\[m^{0j} = -\frac{2sg^{00} e^{\Omega} g^{ia} z_a + \cdots}{(1 + 2s) + \triangle (m)}\]

\[\triangle (m) = (1 + 2s)(g^{00} + 1)(g^{00} - 1) + 2(1 + 2s)e^{\Omega} g^{00} g^{0a} z_a + \cdots\]

\[\triangle^{jk}_{(m)} = 2e^{\Omega} (g^{jk} g^{0a} + 2sg^{0k} g^{aj}) z_a + \cdots\]

\[Z_j \overset{\text{def}}{=} \frac{e^{-\Omega} \partial_j \Phi}{\partial_t \Phi}\]
Theorem

(I.R. & J.S. 2009) Assume that $N \geq 3$ and $0 < c_s^2 < 1/3$. Then there exist $\epsilon_0 > 0$ and $C > 1$ such that for all $\epsilon \leq \epsilon_0$, if $Q_N(0) \leq C^{-1} \epsilon$, then there exists a global future causal geodesically complete solution to the reduced irrotational Euler-Einstein system. Furthermore,

$$Q_N(t) \leq \epsilon$$

holds for all $t \geq 0$. 
Asymptotics of the solution

Theorem

(I.R. & J.S. 2009) Under the assumptions of the global existence theorem, with the additional assumption $N \geq 5$, there exist $q > 0$, a smooth Riemann metric $g_{jk}^{(\infty)}$ with corresponding Christoffel symbols $\Gamma_{ijk}^{(\infty)}$ and inverse $g_{jk}^{(\infty)}$ on $\mathbb{T}^3$, and (time independent) functions $\bar{\partial}\Phi^{(\infty)}$, $\Psi^{(\infty)}$ on $\mathbb{T}^3$ such that

\[
\| e^{-2\Omega} g_{jk}(t, \cdot) - g_{jk}^{(\infty)} \|_{H^N} \leq C\epsilon e^{-qHt}
\]
\[
\| e^{2\Omega} g^{jk}(t, \cdot) - g_{jk}^{(\infty)} \|_{H^N} \leq C\epsilon e^{-qHt}
\]
\[
\| e^{-2\Omega} \partial_t g_{jk}(t, \cdot) - 2\omega g_{jk}^{(\infty)} \|_{H^N} \leq C\epsilon e^{-qHt}
\]
\[
\| g_{0j}(t, \cdot) - H^{-1} g_{ab}^{(\infty)} \Gamma_{ajb}^{(\infty)} \|_{H^{N-3}} \leq C\epsilon e^{-qHt}
\]
\[
\| \partial_t g_{0j}(t, \cdot) \|_{H^{N-3}} \leq C\epsilon e^{-qHt}
\]
Asymptotics cont.

**Theorem**

*(Continued)*

\[ \| g_{00} + 1 \|_{H^N} \leq C \epsilon e^{-qHt} \]
\[ \| \partial_t g_{00} \|_{H^N} \leq C \epsilon e^{-qHt} \]
\[ \| e^{-2\Omega} K_{jk}(t, \cdot) - \omega g_{jk}^{(\infty)} \|_{H^N} \leq C \epsilon e^{-qHt} \]

In the above inequality, \( K_{jk} \) is the second fundamental form of the hypersurface \( t = \text{const} \).

\[ \| e^{w\Omega} \partial_t \Phi(t, \cdot) - \Psi(\infty) \|_{H^{N-1}} \leq C \epsilon e^{-qHt} \]
\[ \| \bar{\partial} \Phi(t, \cdot) - \bar{\partial} \Phi^{(\infty)} \|_{H^{N-1}} \leq C \epsilon e^{-qHt} . \]
Comparison with flat space

Christodoulou’s monograph “The Formation of Shocks in 3-Dimensional Fluids” shows that on the Minkowski space background, shock singularities can form in solutions to the irrotational fluid equation arising from data that are arbitrarily close to that of a uniform quiet state.

Conclusion: Exponentially expanding spacetimes can stabilize irrotational fluids.
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Conclusion: Exponentially expanding spacetimes can stabilize irrotational fluids.
The study of wave equations arising from metrics featuring accelerated expansion is a "very local" problem.

A patching argument can "very likely" be used to allow for many of the topologies considered by Ringström: Unimodular Lie Groups different from $SU_2; \mathbb{H}^3; \mathbb{H}^2 \times \mathbb{R}; \ldots$.
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Future directions

- Other equations of state
  - Sub-exponential expansion rates
  - Non-zero vorticity: forthcoming article
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Thank you
The norms

\( S_{g_{00}+1;N} \equiv e^{q\Omega} \| \partial_t g_{00} \|_{H^N} + e^{(q-1)\Omega} \| \bar{\partial} g_{00} \|_{H^N} + e^{q\Omega} \| g_{00} + 1 \|_{H^N} \)

\( S_{g_{0*};N} \equiv \sum_{j=1}^{3} \left( e^{(q-1)\Omega} \| \partial_t g_{0j} \|_{H^N} + e^{(q-2)\Omega} \| \bar{\partial} g_{0j} \|_{H^N} + e^{(q-1)\Omega} \| g_{0j} \|_{H^N} \right) \)

\( S_{h_{**};N} \equiv \sum_{j,k=1}^{3} \left( e^{q\Omega} \| \partial_t h_{jk} \|_{H^N} + e^{(q-1)\Omega} \| \bar{\partial} h_{jk} \|_{H^N} + \| \bar{\partial} h_{jk} \|_{H^{N-1}} \right) \)

\( S_N \equiv e^{w\Omega} \| \partial_t \Phi - \bar{\Psi}_0 \|_{H^N} + e^{(w-1)\Omega} \| \bar{\partial} \Phi \|_{H^N} \)

\( Q_N \equiv S_{g_{00}+1;N} + S_{g_{0*};N} + S_{h_{**};N} + S_N \)
If $\alpha > 0$, $\beta \geq 0$, and $v$ is a solution to

$$\hat{\square}_g v = \alpha H \partial_t v + \beta H^2 v + F,$$

then we control solutions to this equation using an energy of the form

$$\mathcal{E}^2_{(\gamma, \delta)}[v, \partial v] \overset{\text{def}}{=}$$

$$+ \frac{1}{2} \int \left\{ -g^{00}(\partial_t v)^2 + g^{ab} (\partial_a v)(\partial_b v) - 2\gamma H g^{00} v \partial_t v + \delta H^2 v^2 \right\} d^3 x$$
Energies for $g_{\mu\nu}$

\[
\mathcal{E}^2_{g_{00} + 1; N} \overset{\text{def}}{=} \sum_{|\vec{\alpha}| \leq N} e^{2q \Omega} \mathcal{E}^2_{\gamma_{00}, \delta_{00}}[\partial_{\vec{\alpha}}(g_{00} + 1), \partial(\partial_{\vec{\alpha}}(g_{00} + 1))]
\]

\[
\mathcal{E}^2_{g_{0*}; N} \overset{\text{def}}{=} \sum_{|\vec{\alpha}| \leq N} \sum_{j=1}^{3} e^{2(q-1)\Omega} \mathcal{E}^2_{\gamma_{0*}, \delta_{0*}}[\partial_{\vec{\alpha}}g_{0j}, \partial(\partial_{\vec{\alpha}}g_{0j})]
\]

\[
\mathcal{E}^2_{h_{**}; N} \overset{\text{def}}{=} \sum_{|\vec{\alpha}| \leq N} \left\{ \sum_{j,k=1}^{3} e^{2q \Omega} \mathcal{E}^2_{0,0}[0, \partial(\partial_{\vec{\alpha}}h_{jk})] + \frac{1}{2} \int_{\mathbb{T}^3} c_{\vec{\alpha}} H^2(\partial_{\vec{\alpha}}h_{jk}) \, d^3 x \right\}
\]
Energies for $\partial \Phi$

$$E^2_0 \overset{\text{def}}{=} \frac{1}{2} \int_{\mathbb{T}^3} (e^{w\Omega} \partial_t \Phi - \bar{\Psi}_0)^2 + e^{2w\Omega} m^{jk}(\partial_j \Phi)(\partial_k \Phi) \, d^3 x$$

$$E^2_N \overset{\text{def}}{=} E^2_0$$

$$+ \sum_{1 \leq |\vec{\alpha}| \leq N} \frac{1}{2} \int_{\mathbb{T}^3} e^{2w\Omega} (\partial_t \partial_{\vec{\alpha}} \Phi)^2 + e^{2w\Omega} m^{jk}(\partial_j \partial_{\vec{\alpha}} \Phi)(\partial_k \partial_{\vec{\alpha}} \Phi) \, d^3 x.$$
Equivalence of norms and energies

Lemma

If $Q_N(t)$ is small enough, then the norms and energies are equivalent.
The Euler-Einstein system

With $R = g^{\alpha\beta} R_{\alpha\beta}$, Einstein’s field equations are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}^{(\text{fluid})}$$

$$T_{\mu\nu}^{(\text{fluid})} = (\rho + p) u_{\mu} u_{\nu} + p g_{\mu\nu}$$

$\rho = \text{proper energy density}$, $p = \text{pressure}$, $u = \text{four-velocity}$

$u$ is future-directed, and

$$u_{\alpha} u^{\alpha} = -1$$
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\( \rho = \) proper energy density, \( p = \) pressure, \( u = \) four-velocity

\( u \) is future-directed, and

\[
uu = -1
\]
Consequence of the Bianchi identities
\[ D_\mu \left( R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \right) = 0: \]

\[ D_\mu \, T^{\mu\nu}_{(\text{fluid})} = 0 \]  \hspace{1cm} (1)

Conservation of particle number:
\[ D_\mu (nu^\mu) = 0 \]  \hspace{1cm} (2)

\( n = \) proper number density

(1) and (2) are the relativistic Euler equations.
The relativistic Euler equations

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\[ D_\mu T_{(\text{fluid})}^{\mu\nu} = 0 \] (1)

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\[ D_\mu (n u^\mu) = 0 \] (2)

\( n = \text{proper number density} \)

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\[ D_\mu \left( T^{\mu\nu}_{\text{fluid}} \right) = 0 \quad (1) \]

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(1) and (2) are the relativistic Euler equations.
Equation of state

To close the Euler system, we need more equations.

Fundamental thermodynamic relation:

\[ \rho + p = n \frac{\partial \rho}{\partial n} \]

Our equation of state:

\[ p = c_s^2 \rho \]

\[ 0 < c_s < \sqrt{\frac{1}{3}} \]

\( c_s \) is the speed of sound
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Fluid vorticity

\[ \sqrt{\sigma} \overset{\text{def}}{=} \frac{\rho + p}{n}, \]

where \( \sigma \geq 0 \). \( \sqrt{\sigma} \) is known as the enthalpy per particle.

The fluid vorticity is defined by

\[ \nu_{\mu\nu} \overset{\text{def}}{=} \partial_{\mu}\beta_{\nu} - \partial_{\nu}\beta_{\mu}, \]

where

\[ \beta_{\mu} \overset{\text{def}}{=} -\sqrt{\sigma}u_{\mu}. \]
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\[ \beta_\mu \overset{\text{def}}{=} -\sqrt{\sigma} u_\mu. \]

where \( \sigma \geq 0 \). \( \sqrt{\sigma} \) is known as the enthalpy per particle.
An **irrotational** fluid is one for which

$$\nu_{\mu\nu} \overset{\text{def}}{=} \partial_{\mu} \beta_{\nu} - \partial_{\nu} \beta_{\mu} = 0$$

Local consequence:

$$\beta_{\mu} \overset{\text{def}}{=} -\sqrt{\sigma} u_{\mu} = -\partial_{\mu} \Phi$$

\(\Phi\) is the fluid potential. We think of \(\partial \Phi\) existing globally; \(\Phi\) itself may only exist locally (Poincaré’s lemma). Only \(\partial \Phi\) enters into the equations.
An irrotational fluid is one for which

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Consequence: for irrotational fluids, the Euler equations
\[ D_\mu T^\mu_{\text{(fluid)}} = 0, \quad D_\mu (n u^\mu) = 0 \]
are equivalent to a scalar quasilinear wave equation that is the Euler-Lagrange equation corresponding to a Lagrangian \( \mathcal{L}(\sigma) \):

\[
D_\alpha \left[ \frac{\partial \mathcal{L}}{\partial \sigma} D^\alpha \Phi \right] = 0.
\]

\[
\mathcal{L} = \rho
\]

\[
u_\mu = -\sigma^{-1/2} \partial_\mu \Phi
\]

\[
\sigma = -g^{\alpha\beta}(\partial_\alpha \Phi)(\partial_\beta \Phi)
\]
Scalar formulation of irrotational fluid mechanics

Consequence: for irrotational fluids, the Euler equations $D_\mu T_{\mu \nu}^{(\text{fluid})} = 0, D_\mu (nu_\mu) = 0$ are equivalent to a scalar quasilinear wave equation that is the Euler-Lagrange equation corresponding to a Lagrangian $\mathcal{L}(\sigma)$:

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Corresponding to $\mathcal{L}$, we have the energy-momentum tensor:

$$T^{(\text{scalar})}_{\mu\nu} \stackrel{\text{def}}{=} 2 \frac{\partial \mathcal{L}}{\partial \sigma} (\partial_\mu \Phi)(\partial_\nu \Phi) + g_{\mu\nu} \mathcal{L}$$

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The case $p = c_s^2 \rho$

For $p = c_s^2 \rho$, $s = \frac{1-c_s^2}{2c_s^2}$; $c_s^2 = \frac{1}{2s+1}$, it follows that

$$\mathcal{L} = p = (1 + s)^{-1} \sigma^{s+1}$$

$$T_{\mu\nu}^{(\text{scalar})} = 2\sigma^s (\partial_\mu \Phi)(\partial_\nu \Phi) + g_{\mu\nu} (1 + s)^{-1} \sigma^{s+1}$$

The fluid equation:

$$D_\alpha (\sigma^s D^\alpha \Phi) = 0$$
The case \( p = c_s^2 \rho \)

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\]

The fluid equation:

\[
D_\alpha(\sigma^s D^\alpha \Phi) = 0
\]
To construct a background solution, \( \tilde{g}, \tilde{\Phi} \), we make the ansatz

\[
\tilde{g} = -dt^2 + a(t)^2 \sum_{k=1}^{3} (dx^k)^2,
\]

from which it follows that the only corresponding non-zero Christoffel symbols are

\[
\tilde{\Gamma}^0_{j k} = \tilde{\Gamma}^0_{k j} = \dot{a} \delta_{jk},
\]

\[
\tilde{\Gamma}^k_{j 0} = \tilde{\Gamma}^k_{0 j} = \frac{\dot{a}}{a} \delta^k_j.
\]
Plugging this ansatz into the Euler-Einstein system, we deduce that:

\[
\rho a^{3(1+A^2)} \equiv \bar{\rho} a^{3(1+A^2)} \overset{\text{def}}{=} \kappa_0
\]

\[
\dot{a} = a \sqrt{\frac{\Lambda}{3} + \frac{\rho}{3}} = a \sqrt{\frac{\Lambda}{3} + \frac{\kappa_0}{3 a^{3(1+A^2)}}}
\]

\[\bar{\rho} > 0, \quad \dot{a} = a(0)\]

\[
e^\Omega(t) \overset{\text{def}}{=} a(t) \approx e^{-Ht}, \quad H \overset{\text{def}}{=} \sqrt{\frac{\Lambda}{3}}
\]

\[
\partial \tilde{\Phi} = (\partial_t \tilde{\Phi}, 0, 0, 0)
\]

\[
\partial_t \tilde{\Phi} = \left(\kappa_0 \frac{s + 1}{2s + 1}\right)^{1/(2s+2)} e^{-W\Omega}, \quad W = \frac{3}{1+2s}
\]
Wave coordinates

To hyperbolize the Einstein equations, we use a variant of the wave coordinates:

\[ \Gamma^\nu = \tilde{\Gamma}^\nu = 3\omega \delta^0_\nu \]

\[ \omega \overset{\text{def}}{=} \partial_t \Omega \approx H = \sqrt{\frac{\Lambda}{3}} \]

In wave coordinates, \( \Box_g \phi = 0 \iff \hat{\Box}_g \phi = 3\omega \partial_t \phi \) dissipative term
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\[ \square_g \phi \overset{\text{def}}{=} g^{\alpha \beta} D_\alpha D_\beta \phi \]

\[ \hat{\square}_g \phi \overset{\text{def}}{=} g^{\alpha \beta} \partial_\alpha \partial_\beta \phi \]

In wave coordinates,
\[ \square_g \phi = 0 \iff \hat{\square}_g \phi = 3\omega \partial_t \phi \]

dissipative term
Energy vs. norm comparison

Lemma

If $\mathcal{S}_{g_{00}+1;N} + \mathcal{E}_{g_{0*};N} + \mathcal{E}_{h_{**};N} + S_N$ is sufficiently small, then

\begin{align*}
C^{-1} \mathcal{E}_{g_{00}+1;N} & \leq \mathcal{S}_{g_{00}+1;N} \leq C \mathcal{E}_{g_{00}+1;N} \\
C^{-1} \mathcal{E}_{g_{0*};N} & \leq \mathcal{S}_{g_{0*};N} \leq C \mathcal{E}_{g_{0*};N} \\
C^{-1} \mathcal{E}_{h_{**};N} & \leq \mathcal{S}_{h_{**};N} \leq C \mathcal{E}_{h_{**};N} \\
C^{-1} E_N & \leq S_N \leq CE_N
\end{align*}
Integral inequalities

\[ E_N(t) \leq E_N(t_1) + C \int_{\tau=t_1}^{t} e^{-qH\tau} Q_N(\tau) \, d\tau \]

\[ \mathcal{E}_{g_0+1;N}(t) \leq \mathcal{E}_{g_0+1;N}(t_1) + \int_{\tau=t_1}^{t} -2qH\mathcal{E}_{g_0+1;N}(\tau) + C e^{-qH\tau} Q_N(\tau) \, d\tau \]

\[ \mathcal{E}_{g_0;N}(t) \leq \mathcal{E}_{g_0;N}(t_1) + \int_{\tau=t_1}^{t} -2qH\mathcal{E}_{g_0;N}(\tau) + C e^{-qH\tau} Q_N(\tau) \, d\tau \]

\[ \mathcal{E}_{h_*;N}(t) \leq \mathcal{E}_{h_*;N}(t_1) + \int_{\tau=t_1}^{t} \frac{1}{2} H e^{-qH\tau} \mathcal{E}_{h_*;N}(\tau) + C e^{-qH\tau} Q_N(\tau) \, d\tau. \]