An Overview of Recent Progress on Shock Formation in Three Spatial Dimensions

Jared Speck
with G. Holzegel, S. Klainerman, & W. Wong

Massachusetts Institute of Technology
Theorem (Classic, Sbierski, Wong)

Consider the initial value problem

\[ \Box_m \phi = \mathcal{N}(\phi, \partial \phi, \partial^2 \phi) := \text{Nonlinearity}, \]
\[ \phi|_{\Sigma_0} = \phi, \quad \partial_t \phi|_{\Sigma_0} = \phi_0, \]

on \( \mathbb{R}^{1+3} \), where \( m = \text{diag}(-1, 1, 1, 1) \), \( \Box_m := -\partial_t^2 + \Delta \), \( \Sigma_0 \) is a spacelike hypersurface, and \( (\phi, \phi_0) \) are smooth. Then \( \exists! \) maximal development.

What is the nature of the maximal development?
- Global existence?
- Singularities?
- Cauchy horizon?
Consider the initial value problem

\[ \Box_m \phi = N(\phi, \partial \phi, \partial^2 \phi) := \text{Nonlinearity},\]

\[ \phi|_{\Sigma_0} = \dot{\phi}, \quad \partial_t \phi|_{\Sigma_0} = \ddot{\phi}_0, \]

on \( \mathbb{R}^{1+3} \), where \( m = \text{diag}(-1, 1, 1, 1) \), \( \Box_m := -\partial^2_t + \Delta \), \( \Sigma_0 \) is a spacelike hypersurface, and \((\dot{\phi}, \ddot{\phi}_0)\) are smooth. Then \( \exists! \) maximal development.

What is the nature of the maximal development?

- Global existence?
- Singularities?
- Cauchy horizon?
A maximal development with a singular boundary

\[ \Sigma_0 = \text{spacelike} \]
Burgers’ equation for $\Psi(t, x)$:

$$\partial_t \Psi + \Psi \partial_x \Psi = 0$$

- $\hat{\epsilon} =$ data size; data $:= \Psi(0, x)$

It is famously known that shock singularities form at $T_{\text{Shock}} = O(\hat{\epsilon}^{-1})$; $\Psi$ remains bounded but $|\partial_x \Psi| \rightarrow \infty$

A much less appreciated fact is that:

The singularity is renormalizable. That is, the solution is globally smooth relative to good coordinates.
A model problem in 1D

Burgers’ equation for $\psi(t, x)$:

$$\partial_t \psi + \psi \partial_x \psi = 0$$

- $\hat{\epsilon}$ = data size; data := $\psi(0, x)$
- It is famously known that shock singularities form at $T_{Shock} = \mathcal{O}(\hat{\epsilon}^{-1})$; $\psi$ remains bounded but $|\partial_x \psi| \to \infty$

A much less appreciated fact is that:

The singularity is renormalizable. That is, the solution is globally smooth relative to good coordinates.
A model problem in 1D

Burgers’ equation for $\psi(t, x)$:

$$\partial_t \psi + \psi \partial_x \psi = 0$$

- $\hat{\epsilon} =$ data size; data := $\psi(0, x)$
- It is famously known that shock singularities form at $T_{Shock} = \mathcal{O}(\hat{\epsilon}^{-1})$; $\psi$ remains bounded but $|\partial_x \psi| \to \infty$

A much less appreciated fact is that:

**The singularity is renormalizable.** That is, the solution is globally smooth relative to good coordinates.
“Hiding” the singularity

Characteristics: $x = x(t; u)$ solves

$$\frac{d}{dt} x(t; u) = \psi(t, x(t; u)), \quad x(0; u) = u$$

In characteristic coordinates $(t, u)$, Burgers’ equation is

$$\frac{\partial}{\partial t} \psi(t, u) = 0$$

Thus,

$$\psi(t, u) = F(u)$$
“Hiding” the singularity

Characteristics: \( x = x(t; u) \) solves

\[
\frac{d}{dt} x(t; u) = \psi(t, x(t; u)), \quad x(0; u) = u
\]

In characteristic coordinates \((t, u)\), Burgers’ equation is

\[
\frac{\partial}{\partial t} \psi(t, u) = 0
\]

Thus,

\[
\psi(t, u) = F(u)
\]
Shocks in Burgers’ equation solutions

\[ \mu := \frac{1}{\partial_x u} = 0 \quad u \equiv \text{const} \]

\[ T_{\text{Shock}} = \mathcal{O}(\hat{\epsilon}^{-1}) \]

\[ \hat{\epsilon} = \text{data size} \]
The inverse foliation density $\mu$

Set

\[ \mu := \frac{1}{\partial_x u} \]

Evolution equation for $\mu$:

\[ \frac{\partial}{\partial t} \mu(t, u) = \frac{\partial}{\partial u} \psi(t, u) = F'(u) \]

CHOV relation $\implies \partial_x \psi$ blows up when $\mu \to 0$:

\[ \partial_x \psi = \frac{1}{\mu} \frac{\partial}{\partial u} \psi = \frac{1}{\mu} F' \]
The inverse foliation density $\mu$

Set

$$\mu := \frac{1}{\partial_x u}$$

Evolution equation for $\mu$:

$$\frac{\partial}{\partial t} \mu(t, u) = \frac{\partial}{\partial u} \psi(t, u) = F'(u)$$

CHOV relation $\implies \partial_x \psi$ blows up when $\mu \to 0$:

$$\partial_x \psi = \frac{1}{\mu} \frac{\partial}{\partial u} \psi = \frac{1}{\mu} F'$$
The inverse foliation density $\mu$

Set

$$\mu := \frac{1}{\partial_x u}$$

**Evolution equation for $\mu$:**

$$\frac{\partial}{\partial t} \mu(t, u) = \frac{\partial}{\partial u} \psi(t, u) = F'(u)$$

**CHOV relation** $\implies \partial_x \psi$ blows up when $\mu \to 0$:

$$\partial_x \psi = \frac{1}{\mu} \frac{\partial}{\partial u} \psi = \frac{1}{F'}$$
The 1D theory

There are a huge number of related results 1D:

- Challis (1848)
- ... 
- Zabusky (1962)
- Lax (1964)
- Keller and Ting (1966)
- John (1974)
- Klainerman and Majda (1980)
- Majda (1984)
- ...
Theorem (John 1980)

In $\mathbb{R}^{1+3}$, consider nontrivial initially smooth solutions to

$$\square_m \phi = \begin{cases} -\left( \partial_t \phi \right)^2, \\ -\left( \partial_t \phi \right) \partial^2_t \phi \end{cases},$$

$$\phi|_{t=0} = \phi, \quad \partial_t \phi|_{t=0} = \dot{\phi}_0,$$

where $m = \text{diag}(-1, 1, 1, 1)$ and $\square_m := -\partial_t^2 + \Delta$. Then $\phi$ breaks down for some unknown reason.

A different kind of non-constructive blow-up proof for the Euler equations was provided by Sideris and later by Guo+Tahvildar-Zadeh.
Non-constructive breakdown in 3D

Theorem (John 1980)

In $\mathbb{R}^{1+3}$, consider nontrivial initially smooth solutions to

$$\Box_m \phi = \begin{cases} -(\partial_t \phi)^2, \\ -(\partial_t \phi)\partial_t^2 \phi \end{cases},$$

$$\phi|_{t=0} = \dot{\phi}, \quad \partial_t \phi|_{t=0} = \dot{\phi}_0,$$

where $m = \text{diag}(-1, 1, 1, 1)$ and $\Box_m : = -\partial_t^2 + \Delta$. Then $\phi$ breaks down for some unknown reason.

- A different kind of non-constructive blow-up proof for the Euler equations was provided by Sideris and later by Guo + Tahvildar-Zadeh.
Null condition in 3D

In the previous examples, S. Klainerman’s classic null condition fails; there are quadratic terms proportional to

\[(\nabla_{L_{\text{Flat}}} \phi)^2, \quad \nabla_{L_{\text{Flat}}} \phi \cdot \nabla^2_{L_{\text{Flat}}} \phi\]
Theorem (Christodoulou, Klainerman 1984)

For data of small size $\hat{\epsilon}$, solutions to

$$\square_m \phi = Q_1(\partial \phi, \partial \phi) + Q_2(\partial^2 \phi, \partial \phi)$$

exist globally when the $Q_i$ verify the classic null condition.

Example: $Q_1(\partial \phi, \partial \phi) = (\partial_t \phi)^2 - (\partial_r \phi)^2 = \nabla_{L_{Flat}} \phi \cdot \nabla_{L_{Flat}} \phi$

Key ingredient: linear solutions $\square_m \phi = 0$ disperse:

$$|\nabla_{L_{Flat}} \phi| \lesssim \hat{\epsilon} \frac{1}{1+t},$$

$$|\nabla_{L_{Flat}} \phi|, |\nabla \phi| \lesssim \hat{\epsilon} \frac{1}{(1+t)^2}.$$
Relevance of null condition

**Theorem (Christodoulou, Klainerman 1984)**

For data of small size $\hat{\epsilon}$, solutions to

$$\Box_m \phi = Q_1(\partial \phi, \partial \phi) + Q_2(\partial^2 \phi, \partial \phi)$$

exist globally when the $Q_i$ verify the classic null condition.

Example: $Q_1(\partial \phi, \partial \phi) = (\partial_t \phi)^2 - (\partial_r \phi)^2 = \nabla_{L_{\text{Flat}}} \phi \cdot \nabla_{L_{\text{Flat}}} \phi$

Key ingredient: linear solutions $\Box_m \phi = 0$ disperse:

$$|\nabla_{L_{\text{Flat}}} \phi| \lesssim \hat{\epsilon} \frac{1}{1 + t},$$

$$|\nabla_{L_{\text{Flat}}} \phi|, |\nabla \phi| \lesssim \hat{\epsilon} \frac{1}{(1 + t)^2}$$
John’s proof of spherically symmetric blow-up

**Theorem (John 1985)**

In 3D, consider nontrivial spherically symmetric data of small size $\epsilon$ for

$$-\partial_t^2 \phi + (1 + \partial_t \phi) \Delta \phi = 0$$

Then $\nabla_L \partial_r \phi$ blows up at $T_{\text{Lifespan}} = O(\exp(c\epsilon^{-1}))$. $\phi$ and its first rectangular derivatives remain bounded.

Blow-up follows from a Riccati inequality for $w \sim r \nabla_L \partial_r \phi$ along characteristics:

$$\frac{d}{dt} w \geq c \frac{1}{1 + t} w^2 + \text{Error}$$
Theorem (John 1985)

In 3D, consider nontrivial spherically symmetric data of small size $\epsilon$ for

$$-\partial_t^2 \phi + (1 + \partial_t \phi) \Delta \phi = 0$$

Then $\nabla L \partial_r \phi$ blows up at $T_{\text{Lifespan}} = O(\exp(c \epsilon^{-1})))$. $\phi$ and its first rectangular derivatives remain bounded.

Blow-up follows from a Riccati inequality for $w \sim r \nabla L \partial_r \phi$

along characteristics:

$$\frac{d}{dt} w \geq c \frac{1}{1 + t} w^2 + \text{Error}$$
Main difficulties away from symmetry

Away from symmetry, the term Error in

$$\frac{d}{dt} w \geq c \frac{1}{1 + t} w^2 + \text{Error}$$

depends on the angular Laplacian $\Delta \phi$.

- Angular derivatives can only be controlled with energy estimates.
- Suitable energy estimates showing that $\nabla \phi = \text{Error}$ are very hard to derive near the blow-up points.
- The Riccati inequality approach is not precise enough to work.
Main difficulties away from symmetry

Away from symmetry, the term Error in

\[ \frac{d}{dt} w \geq c \frac{1}{1 + t} w^2 + \text{Error} \]

depends on the angular Laplacian \( \Delta \phi \).

- Angular derivatives can only be controlled with energy estimates.

- Suitable energy estimates showing that \( \nabla \phi = \text{Error} \) are very hard to derive near the blow-up points.

- The Riccati inequality approach is not precise enough to work.
Theorem (Alinhac 1990s-early 2000s)

In 3D, consider nontrivial data of small size $\tilde{\epsilon}$ for

\[
(g^{-1})^{\alpha\beta}(\partial\phi)\partial_\alpha \partial_\beta \phi = 0, \quad \text{(Einstein summation)}
\]

\[
g_{\alpha\beta}(\partial\phi) = m_{\alpha\beta} + g^{\text{Pert}}_{\alpha\beta}(\partial\phi), \quad g^{\text{Pert}}_{\alpha\beta}(\partial\phi = 0) = 0
\]

Assume that the null condition fails. Then under a non-degeneracy assumption on the data:

- $\exists$ exactly one blow-up point at $T_{\text{Shock}} = O(\exp(c\tilde{\epsilon}^{-1}))$.
- Singularity can be partially renormalized.
- Blow-up is tied to intersecting characteristics.

Extensions by Huicheng, Bingbing, Witt
Alinhac’s shock formation results

Theorem (Alinhac 1990s-early 2000s)

In 3D, consider nontrivial data of small size $\epsilon$ for

$$(g^{-1})^{\alpha\beta}(\partial\phi)\partial_\alpha\partial_\beta\phi = 0,$$  
(Einstein summation)

$$g_{\alpha\beta}(\partial\phi) = m_{\alpha\beta} + g^{\text{Pert}}_{\alpha\beta}(\partial\phi),$$

$$g^{\text{Pert}}_{\alpha\beta}(\partial\phi = 0) = 0$$

Assume that the null condition fails. Then under a non-degeneracy assumption on the data:

$\exists$ exactly one blow-up point at $T_{\text{Shock}} = O(\exp(c\epsilon^{-1}))$.

Singularity can be partially renormalized.

Blow-up is tied to intersecting characteristics.

Extensions by Huicheng, Bingbing, Witt
The region studied by Alinhac
In his famous book of 2007, Christodoulou (and with Miao in 2013) proved a related small-data blow-up result. His proof yielded valuable new information that is inaccessible via Alinhac’s approach.

He studied perturbations of the solution $\phi \equiv t$ to all Euler-Lagrange PDEs from irrotational fluid mechanics.

Example equations are the following with $s > 1$:

$$\partial_\alpha \left\{ \frac{\partial \sigma^2}{\partial (\partial_\alpha \phi)} \right\} = 0, \quad \sigma := - (m^{-1})^{\kappa \lambda} \partial_\kappa \phi \partial_\lambda \phi$$

Singularities form $\iff$ characteristics intersect.

To generate shocks, he made assumptions on

$$\int_{S_{d,u}} r \left\{ (\partial_t \phi - 1) - c \partial_r \phi \right\} \, d\sigma + \int_{\Sigma^1_{d,u}} 2(\partial_t \phi - 1) - c \partial_r \phi \, d^3x$$
Christodoulou’s shock formation results in 3D

In his famous book of 2007, Christodoulou (and with Miao in 2013) proved a related small-data blow-up result. His proof yielded valuable new information that is inaccessible via Alinhac’s approach.

He studied perturbations of the solution $\phi \equiv t$ to all Euler-Lagrange PDEs from irrotational fluid mechanics.

Example equations are the following with $s > 1$:

$$\partial_\alpha \left\{ \frac{\partial \sigma^2}{\partial (\partial_\alpha \phi)} \right\} = 0, \quad \sigma := -(m^{-1})^{s \lambda} \partial_\kappa \phi \partial_\chi \phi$$

Singularities form $\iff$ characteristics intersect.

To generate shocks, he made assumptions on

$$\int_{S_{t,\nu}} r \left\{ (\partial_t \phi - 1) - c \partial_r \phi \right\} \, d\sigma + \int_{\Sigma_{t,\nu}} 2(\partial_t \phi - 1) - c \partial_r \phi \, d^3x$$
In his famous book of 2007, Christodoulou (and with Miao in 2013) proved a related small-data blow-up result. His proof yielded valuable new information that is inaccessible via Alinhac’s approach.

He studied perturbations of the solution $\phi \equiv t$ to all Euler-Lagrange PDEs from irrotational fluid mechanics.

Example equations are the following with $s > 1$:

$$\partial_\alpha \left\{ \frac{\partial \sigma^s}{\partial (\partial_\alpha \phi)} \right\} = 0, \quad \sigma := -(m^{-1})^{\kappa \lambda} \partial_\kappa \phi \partial_\lambda \phi$$

Singularity form $\iff$ characteristics intersect.

To generate shocks, he made assumptions on

$$\int_{S_{0,t}} \left( (\partial_t \phi - 1) - c \partial_r \phi \right) \, d\sigma + \int_{\Sigma_0^t} 2(\partial_t \phi - 1) - c \partial_r \phi \, d^3x$$
In his famous book of 2007, Christodoulou (and with Miao in 2013) proved a related small-data blow-up result. His proof yielded valuable new information that is inaccessible via Alinhac’s approach.

He studied perturbations of the solution $\phi \equiv t$ to all Euler-Lagrange PDEs from irrotational fluid mechanics.

Example equations are the following with $s > 1$:

$$\partial_\alpha \left\{ \frac{\partial \sigma^s}{\partial (\partial_\alpha \phi)} \right\} = 0, \quad \sigma := -(m^{-1})^{\kappa \lambda} \partial_\kappa \phi \partial_\lambda \phi$$

Singularities form $\iff$ characteristics intersect.
In his famous book of 2007, Christodoulou (and with Miao in 2013) proved a related small-data blow-up result. His proof yielded valuable new information that is inaccessible via Alinhac’s approach.

He studied perturbations of the solution $\phi \equiv t$ to all Euler-Lagrange PDEs from irrotational fluid mechanics.

Example equations are the following with $s > 1$:

$$
\partial_{\alpha} \left\{ \frac{\partial \sigma^s}{\partial (\partial_{\alpha} \phi)} \right\} = 0, \quad \sigma := -(m^{-1})^{\kappa\lambda} \partial_{\kappa} \phi \partial_{\lambda} \phi
$$

Singularities form $\iff$ characteristics intersect.

To generate shocks, he made assumptions on

$$
\int_{S_{0,\nu}} r \left\{ (\partial_t \phi - 1) - c \partial_r \phi \right\} \, d\sigma + \int_{\Sigma^u_{0}} 2(\partial_t \phi - 1) - c \partial_r \phi \, d^3x
$$
New information in Christodoulou's proof

Figure: The maximal development
**Theorem (G.H., S.K., J.S., W.W.)**

Consider the initial value problem

\[
\square g(\psi) \psi := (g^{-1})^{\alpha\beta} (\psi) \partial_\alpha \partial_\beta \psi - \Gamma^\alpha (\psi, \partial \psi) \partial_\alpha \psi = 0, \\
\psi|_{t=0} = \dot{\psi}, \quad \partial_t \psi|_{t=0} = \dot{\psi}_0
\]

where \((\dot{\psi}, \dot{\psi}_0)\) are compactly supported and of small size \(\dot{\epsilon}\). Assume that

- \(g_{\alpha\beta}(\psi) = m_{\alpha\beta} + g^{\text{Pert}}_{\alpha\beta}(\psi), \ g^{\text{Pert}}_{\alpha\beta}(0) = 0\)
- **Classic null condition fails**

We provide a complete description of the dynamics.

- Singularity \(\iff\) characteristics intersect.
- Rescaling \((\dot{\psi}, \dot{\psi}_0) \to \lambda (\dot{\psi}, \dot{\psi}_0)\) for small \(\lambda\) always leads to a shock at \(T_{\text{Shock}} \sim \exp(c \dot{\epsilon}^{-1})\).
Theorem (G.H., S.K., J.S., W.W.)

Consider the initial value problem

\[ \square g(\psi) \psi := (g^{-1})^{\alpha\beta}(\psi) \partial_\alpha \partial_\beta \psi - \Gamma^\alpha(\psi, \partial_\psi) \partial_\alpha \psi = 0, \]
\[ \psi|_{t=0} = \hat{\psi}, \quad \partial_t \psi|_{t=0} = \hat{\psi}_0 \]

where \((\hat{\psi}, \hat{\psi}_0)\) are compactly supported and of small size \(\hat{\epsilon}\). Assume that

- \(g_{\alpha\beta}(\psi) = m_{\alpha\beta} + g^{\text{Pert}}_{\alpha\beta}(\psi), \quad g^{\text{Pert}}_{\alpha\beta}(0) = 0\)
- Classic null condition fails

We provide a complete description of the dynamics.

- Singularity \(\iff\) characteristics intersect.
- Rescaling \((\hat{\psi}, \hat{\psi}_0) \to \lambda(\hat{\psi}, \hat{\psi}_0)\) for small \(\lambda\) always leads to a shock at \(T_{\text{Shock}} \sim \exp(c \hat{\epsilon}^{-1})\).
Corollary (G.H., S.K., J.S., W.W.)

The same result holds for Alinhac’s equations

\[
(g^{-1})^{\alpha\beta} (\partial \phi) \partial_\alpha \partial_\beta \phi = 0,
\]

\[
g_{\alpha\beta} = m_{\alpha\beta} + g_{\alpha\beta}^{\text{Pert}} (\partial \phi), \quad g_{\alpha\beta}^{\text{Pert}} (\partial \phi = 0) = 0
\]

Remark

Miao-Yu recently obtained a related large-data blow-up result for

\[
- \left[ 1 + (\partial_1 \phi)^2 \right] \partial_1^2 \phi + \Delta \phi = 0
\]
Corollary (G.H., S.K., J.S., W.W.)

The same result holds for Alinhac’s equations

\[(g^{-1})^{\alpha\beta}(\partial \phi)\partial_\alpha \partial_\beta \phi = 0,\]

\[g_{\alpha\beta} = m_{\alpha\beta} + g^{\text{Pert}}_{\alpha\beta}(\partial \phi), \quad g^{\text{Pert}}_{\alpha\beta}(\partial \phi = 0) = 0\]

Remark

Miao-Yu recently obtained a related large-data blow-up result for

\[-\left\{ 1 + (\partial_t \phi)^2 \right\} \partial^2_t \phi + \Delta \phi = 0\]
The three main features of the proof

- The singularity is renormalizable, just like in Burgers’ equation!

- Unlike in 1D, basic existence theory requires estimates for higher derivatives.

- Moreover, the first statement is false for the high derivatives: They are not renormalizable! This leads to severe complications!

We must construct a sharp geometric framework to see the renormalizability and to overcome the difficulties.
The three main features of the proof

- The singularity is renormalizable, just like in Burgers’ equation!
- Unlike in 1D, basic existence theory requires estimates for higher derivatives.
- Moreover, the first statement is false for the high derivatives: They are not renormalizable! This leads to severe complications!

We must construct a sharp geometric framework to see the renormalizability and to overcome the difficulties.
The three main features of the proof

- The singularity is renormalizable, just like in Burgers’ equation!
- Unlike in 1D, basic existence theory requires estimates for higher derivatives.
- Moreover, the first statement is false for the high derivatives: They are not renormalizable! This leads to severe complications!

We must construct a sharp geometric framework to see the renormalizability and to overcome the difficulties.
The three main features of the proof

- The singularity is renormalizable, just like in Burgers’ equation!
- Unlike in $1D$, basic existence theory requires estimates for higher derivatives.
- Moreover, the first statement is false for the high derivatives: They are not renormalizable! This leads to severe complications!

We must construct a sharp geometric framework to see the renormalizability and to overcome the difficulties.
Eikonal function

Most important proof ingredient is the eikonal function:

\[
\mathcal{D}u \cdot \mathcal{D}u := (g^{-1})^{\alpha \beta} (\Psi) \partial_\alpha u \partial_\beta u = 0,
\]
\[
\partial_t u > 0, \quad u|_{t=0} = 1 - r,
\]

- Truncated level sets are null cone pieces \( C_u^t \)
- Play a critical role in many delicate local and global results for wave equations, especially Einstein’s equations (starting with the Christodoulou-Klainerman proof of the stability of Minkowski spacetime)
Fundamental geometric constructions

- $\mu^{-1} := -D_t \cdot D_u$, shock $\iff \mu \to 0$
- $L^\nu := -\mu D^\nu u$, $L \sim \partial_t + \partial_r$
- $\tilde{R} u = 1$, $g(\tilde{R}, \tilde{R}) = \mu^2$, shock $\iff \tilde{R} \to 0$
- $X_1, X_2$ are local vectorfields on $S_{t,u} = \Sigma_t \cap C^t_u$
The system of equations

\[ \square_{g(\psi)} \psi = 0, \]

\[ L \mu = \frac{1}{2} G_{LL} \ddot{R} \psi + \text{Error}(L \psi, \nabla \psi, \ldots), \]

\[ LL^i = \text{Error}(L \psi, \nabla \psi, \ldots) \]

- Last two equations are \( \sim Du \cdot Du = 0 \)
- \( \nabla := \) (true) angular derivatives
- \( L = \frac{\partial}{\partial t} \bigg|_{u, \theta} \)
- \( \ddot{R} = \frac{\partial}{\partial u} \bigg|_{t, \theta} + \) angular error
- \( G_{LL} := \frac{\partial}{\partial \psi} g_{\alpha \beta}(\psi) L^\alpha L^\beta \)
- Classic null condition fails \( \implies G_{LL} \sim f(\psi) \neq 0 \)
The system of equations

\[ \Box_{g(\psi)} \psi = 0, \]
\[ L\mu = \frac{1}{2} G_{LL} \ddot{\mathcal{R}} \psi + \text{Error}(L\psi, \nabla\psi, \cdots), \]
\[ LL^i = \text{Error}(L\psi, \nabla\psi, \cdots) \]

- Last two equations are \( \sim \mathcal{D}u \cdot \mathcal{D}u = 0 \)
- \( \nabla := \) (true) angular derivatives
- \( L = \frac{\partial}{\partial t} |_{u,\vartheta} \)
- \( \ddot{\mathcal{R}} = \frac{\partial}{\partial u} |_{t,\vartheta} + \) angular error
- \( G_{LL} := \frac{d}{d\psi} g_{\alpha\beta}(\psi) L^\alpha L^\beta \)
- Classic null condition fails \( \implies G_{LL} \sim f(\vartheta) \neq 0 \)
Behavior of lower-order derivatives

• $\hat{\epsilon} = \text{data size}$

Bootstrap “linear” behavior relative to the rescaled frame

Relative to the rescaled frame, we should see the dispersive behavior of linear waves:

\[
|L\psi|, |\nabla\psi| \lesssim \hat{\epsilon} \frac{1}{(1 + t)^2},
\]
\[
|\psi|, |\ddot{R}\psi| \lesssim \hat{\epsilon} \frac{1}{1 + t}
\]

• Can prove: $\ddot{R}\psi \sim f_{\text{Data}}(u, \vartheta) \frac{1}{1 + t}$, $f_{\text{Data}} = O(\hat{\epsilon}) \neq 0$

• Captures the renormalizability of the singularity
• Is essential for controlling error terms
Behavior of lower-order derivatives

- $\epsilon = \text{data size}$

Bootstrap “linear” behavior relative to the rescaled frame

Relative to the rescaled frame, we should see the dispersive behavior of linear waves:

\[
|L\psi|, |\nabla \psi| \lesssim \epsilon \frac{1}{(1 + t)^2},
\]
\[
|\psi|, |\tilde{R}\psi| \lesssim \epsilon \frac{1}{1 + t}
\]

Can prove: $\tilde{R}\psi \sim f_{Data}(u, \nu) \frac{1}{1 + t}$, $f_{Data} = O(\epsilon) \neq 0$

- Captures the renormalizability of the singularity
- Is essential for controlling error terms
Behavior of lower-order derivatives

\* \( \hat{\epsilon} = \text{data size} \)

Bootstrap “linear” behavior relative to the *rescaled* frame

Relative to the rescaled frame, we should see the dispersive behavior of linear waves:

\[
|L\psi|, |\nabla\psi| \lesssim \hat{\epsilon} \frac{1}{(1 + t)^2},
\]

\[
|\psi|, |\ddot{R}\psi| \lesssim \hat{\epsilon} \frac{1}{1 + t}
\]

Can prove: \( \ddot{R}\psi \sim f_{\text{Data}}(u, v) \frac{1}{1 + t}, \quad f_{\text{Data}} = O(\hat{\epsilon}) \neq 0 \)

Captures the renormalizability of the singularity

Is essential for controlling error terms
Strong null condition

Set \( \varrho(t, u) := 1 - u + t \approx 1 + t \). Then \( \mu \Box g \Psi = 0 \) is:

\[
L \left\{ \mu L(\varrho \Psi) + 2 \tilde{R}(\varrho \Psi) \right\} = \varrho \mu \Delta \Psi + \text{Error}
\]

- Key structure: Error consists of quadratic and higher terms, where \((\tilde{R} \Psi)^2, (\tilde{R} \Psi)^3\), etc. are not present.
- This is a nonlinear analog of the classic null condition.

“Strong null condition”
Strong null condition

Set $\varrho(t, u) := 1 - u + t \approx 1 + t$. Then $\mu \Box_g \psi = 0$ is:

$$L \left\{ \mu L(\varrho \psi) + 2 \dot{R}(\varrho \psi) \right\} = \varrho \mu \Delta \psi + \text{Error}$$

- Key structure: Error consists of quadratic and higher terms, where $(\dot{R} \psi)^2$, $(\dot{R} \psi)^3$, etc. are not present.

- This is a nonlinear analog of the classic null condition.
  “Strong null condition”
Heuristics behind $\mu \to 0$

\[
L\mu(t, u, \vartheta) \sim \tilde{R}\psi(t, u, \vartheta)
\]

From $\tilde{R}\psi(t, u, \vartheta) \sim \frac{1}{1+t} f_{\text{Data}}(u, \vartheta)$, we obtain:

\[
L\mu(t, u, \vartheta) \sim \frac{1}{1+t} f_{\text{Data}}(u, \vartheta)
\]

Integrating along $L = \frac{\partial}{\partial t}$ and using $\mu|_{t=0} \sim 1$, we conclude:

\[
\mu(t, u, \vartheta) \sim 1 + \ln(1 + t) f_{\text{Data}}(u, \vartheta)
\]

Hence, $\mu \to 0$ at time $T_{\text{Shock}} \sim \exp \left\{ \min_{u, \vartheta} \left( \frac{1}{f_{\text{Data}}(u, \vartheta)} \right) \right\}$
Heuristics behind $\mu \rightarrow 0$

\[ L_{\mu}(t, u, \vartheta) \sim \tilde{R}\psi(t, u, \vartheta) \]

From $\tilde{R}\psi(t, u, \vartheta) \sim \frac{1}{1+t} f_{Data}(u, \vartheta)$, we obtain:

\[ L_{\mu}(t, u, \vartheta) \sim \frac{1}{1 + t} f_{Data}(u, \vartheta) \]

Integrating along $L = \frac{\partial}{\partial t}$ and using $\mu|_{t=0} \sim 1$, we conclude:

\[ \mu(t, u, \vartheta) \sim 1 + \ln(1 + t) f_{Data}(u, \vartheta) \]

Hence, $\mu \rightarrow 0$ at time $T_{Shock} \sim \exp \left\{ |\min_{u, \vartheta} \left( \frac{1}{f_{Data}(u, \vartheta)} \right) | \right\}$
Heuristics behind $\mu \to 0$

$$L_\mu(t, u, \vartheta) \sim R_\psi(t, u, \vartheta)$$

From $R_\psi(t, u, \vartheta) \sim \frac{1}{1+t} f_{Data}(u, \vartheta)$, we obtain:

$$L_\mu(t, u, \vartheta) \sim \frac{1}{1+t} f_{Data}(u, \vartheta)$$

Integrating along $L = \frac{\partial}{\partial t}$ and using $\mu|_{t=0} \sim 1$, we conclude:

$$\mu(t, u, \vartheta) \sim 1 + \ln(1 + t) f_{Data}(u, \vartheta)$$

Hence, $\mu \to 0$ at time $T_{Shock} \sim \exp \left\{ \min_{u, \vartheta} \left( \frac{1}{f_{Data}(u, \vartheta)} \right) \right\}$
Justifying the heuristics

We need to show that

- Error terms are small
- Solution survives until shock

**Big difficulty:** only $L^2$ regularity is propagated at top order.

Two main ingredients:

- A priori $L^2$ estimates for $\Psi, \mu, L^l$
  - hard at top order!
- A priori $C^k$ estimates for $\Psi, \mu, L^l$
Justifying the heuristics

We need to show that

- Error terms are small
- Solution survives until shock

**Big difficulty:** only $L^2$ regularity is propagated at top order.

Two main ingredients:

- A priori $L^2$ estimates for $\psi, \mu, L^i$
  - hard at top order!
- A priori $C^k$ estimates for $\psi, \mu, L^i$
Main a priori $L^2$ estimate hierarchy (hard!)

Let $\hat{\epsilon} := \text{data size}$, $\mu_\star(t, u) := \min\{1, \min_{\Sigma_t} \mu\}$, $E_M(t, u) := L^2(\Sigma_\mu) \text{-based energy of } M\text{ derivatives of } \Psi$. Then, ignoring $\ln t$ factors at top-order, we have:

- $E_{24}(t, u) \lesssim \hat{\epsilon}^2 \mu_\star^{-17.5}(t, u)$
- $E_{23}(t, u) \lesssim \hat{\epsilon}^2 \mu_\star^{-15.5}(t, u)$
- $\ldots$
- $E_{16}(t, u) \lesssim \hat{\epsilon}^2 \mu_\star^{-1.5}(t, u)$
- $E_{15}(t, u) \lesssim \hat{\epsilon}^2$
- $\ldots$
- $E_0(t, u) \lesssim \hat{\epsilon}^2$

We recover $C^k$ bounds for $\Psi$ via Sobolev.

We recover $C^k$ bounds for $\mu, L^I$ from the eikonal equation.
Main a priori $L^2$ estimate hierarchy (hard!)

Let $\hat{\epsilon} := \text{data size}$, $\mu_\star(t, u) := \min\{1, \min_{\Sigma_u} \mu\}$, $E_M(t, u) := L^2(\Sigma^u_t)$—based energy of $M$ derivatives of $\Psi$. Then, ignoring $\ln t$ factors at top-order, we have:

- $E_{24}(t, u) \lesssim \hat{\epsilon}^2 \mu_\star^{-17.5}(t, u)$
- $E_{23}(t, u) \lesssim \hat{\epsilon}^2 \mu_\star^{-15.5}(t, u)$
- \ldots
- $E_{16}(t, u) \lesssim \hat{\epsilon}^2 \mu_\star^{-1.5}(t, u)$
- $E_{15}(t, u) \lesssim \hat{\epsilon}^2$
- \ldots
- $E_0(t, u) \lesssim \hat{\epsilon}^2$

We recover $C^k$ bounds for $\Psi$ via Sobolev

We recover $C^k$ bounds for $\mu$, $L^i$ from the eikonal equation
Energy estimate (divergence theorem) region

\[ \int_{\Sigma_t^u} \mathcal{P}_1[\partial \psi, \partial \psi] + \int_{C_t^u} \mathcal{P}_2[\partial \psi, \partial \psi] \]

\[ = \int_{\Sigma_0^u} \mathcal{P}_1[\partial \psi, \partial \psi] + \int_{\mathcal{M}_{t,u}} \mathcal{Q}[\partial \psi, \partial \psi] \]

\[ + \int_{\mathcal{M}_{t,u}} (\Box g(\psi) \psi) \left\{ (1 + \mu) L \psi + 2 \tilde{R} \psi \right\} \]
Energies and fluxes feature $\mu$ weights

\[ E_0(t, u) \sim \int_{\Sigma_t} \mu |L \psi|^2 + |\dot{R} \psi|^2 + \mu |\nabla \psi|^2, \]

\[ F_0(t, u) \sim \int_{C_{\tilde{u}} t} |L \psi|^2 + \mu |\nabla \psi|^2, \]

Similarly,

\[ E_1(t, u) \sim \int_{\Sigma_t} \mu |LZ \psi|^2 + |\dot{R}Z \psi|^2 + \mu |\nabla Z \psi|^2, \]

e.tc., where the $Z$ are vectorfields depending on $\partial u, \psi, \partial \psi$
Energies and fluxes feature $\mu$ weights

\[
\mathcal{E}_0(t, u) \sim \int_{\Sigma_t} \mu |L\psi|^2 + |\ddot{R}\psi|^2 + \mu |\nabla\psi|^2,
\]
\[
\mathcal{F}_0(t, u) \sim \int_{C_u^t} |L\psi|^2 + \mu |\nabla\psi|^2
\]

Similarly,

\[
\mathcal{E}_1(t, u) \sim \int_{\Sigma_t^u} \mu |LZ\psi|^2 + |\ddot{R}Z\psi|^2 + \mu |\nabla Z\psi|^2,
\]

etc., where the $Z$ are vectorfields depending on $\partial u$, $\psi$, $\partial \psi$
Top-order degeneracy

Why is the top-order estimate so degenerate?

Because the best a priori inequality we are able to obtain is:

$$\mathbb{E}_{24}(t, u) \leq \varepsilon^2 + B \int_{t'}^t \left( \sup_{\Sigma'} \left| \frac{\partial}{\partial t} \ln \mu \right| \right) \mathbb{E}_{24}(t', u) \, dt' + \cdots$$

$$B \sim 10$$

The key difficult factor $\sup_{\Sigma'} \left| \frac{\partial}{\partial t} \ln \mu \right|$ is connected to avoiding derivative loss in $u$.

- Caricature estimate: if $\mu$ were a function of only $t$, then
  Gronwall $\implies \mathbb{E}_{24}(t, u) \lesssim \varepsilon^2 \mu^{-B} + \cdots$

- The correct Gronwall-type estimate is hard to prove because $\mu = \mu(t, u, \psi)$
Top-order degeneracy

Why is the top-order estimate so degenerate?

Because the best a priori inequality we are able to obtain is:

\[
\mathcal{E}_{24}(t, u) \leq \hat{c}^2 + B \int_{t'=0}^{t} \left( \sup_{\Sigma_{t'}} \left| \frac{\partial}{\partial t} \ln \mu \right| \right) \mathcal{E}_{24}(t', u) \, dt' + \cdots
\]

\[B \sim 10\]

The key difficult factor \(\sup_{\Sigma_{t'}} \left| \frac{\partial}{\partial t} \ln \mu \right|\) is connected to avoiding derivative loss in \(u\).

- Caricature estimate: if \(\mu\) were a function of only \(t\), then Gronwall \(\implies\) \(\mathcal{E}_{24}(t, u) \lesssim \hat{c}^2 \mu^{-B} + \cdots\)
- The correct Gronwall-type estimate is hard to prove because \(\mu = \mu(t, u, \psi)\)
Top-order degeneracy

Why is the top-order estimate so degenerate?

Because the best a priori inequality we are able to obtain is:

\[
E_{24}(t, u) \leq \hat{c}^2 + B \int_{t'}^{t} \left( \sup_{t'} \left| \frac{\partial}{\partial t} \ln \mu \right| \right) E_{24}(t', u) \, dt' + \cdots
\]

\[B \sim 10\]

The key difficult factor \(\sup_{t'} \left| \frac{\partial}{\partial t} \ln \mu \right|\) is connected to avoiding derivative loss in \(u\).

- Caricature estimate: if \(\mu\) were a function of only \(t\), then
  Gronwall \(\implies E_{24}(t, u) \lesssim \hat{c}^2 \mu^{-B} + \cdots\)

- The correct Gronwall-type estimate is hard to prove because \(\mu = \mu(t, u, \vartheta)\)
Open problems

Important open problems: proving shock formation for systems with multiple characteristics

Figure: System with two characteristics

- Cosmology (Euler-Einstein)
- Magnetohydrodynamics
- Elasticity
- Nonlinear electromagnetism
- Crystal optics
Latest preprints of the book and survey article are available at:
http://math.mit.edu/ jspeck/publications.html