Geometric Methods in Hyperbolic PDEs

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Unifying mathematical themes

Many physical phenomena are modeled by systems of hyperbolic PDEs.

Keywords associated with hyperbolic PDEs:

- Initial value problem formulation (the “evolution problem”)
- Finite speed of propagation
- Energy estimates

Goal: Show how geometrically motivated techniques can help us extract information about solutions.
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**Goal:** Show how geometrically motivated techniques can help us extract information about solutions.
The Einstein equations

\[ M \overset{\text{def}}{=} 1 + 3 \text{ dimensional manifold} \]

\[ \begin{align*}
Ric_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= T_{\mu\nu} \\
\text{Matter equations} &= 0
\end{align*} \]

- \( g_{\mu\nu} \) = Lorentzian metric of signature \((-,+,+,++)\)
- \( Ric_{\mu\nu} = Ric_{\mu\nu}(g, \partial g, \partial^2 g) \) = Ricci tensor
- \( R = (g^{-1})^{\alpha\beta} Ric_{\alpha\beta} \) = scalar curvature
- \( T_{\mu\nu} \) = energy-momentum tensor of the matter
- Convention: \( X^\nu \overset{\text{def}}{=} (g^{-1})^{\nu\alpha} X_\alpha \), etc.
Euler-Einstein equations with $\Lambda > 0$

\[
\Lambda > 0 \sim \text{“Dark Energy”}
\]

\[
Ric_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}
\]

\[
\nabla_\alpha T^{\alpha\mu} = 0
\]

Euler equations $\leftrightarrow$ Cosmology

- $T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} + pg_{\mu\nu}$
- $g_{\alpha\beta} u^{\alpha} u^{\beta} = -1$
- $u^{0} > 0$ (future-directed)
- $p = c_s^2 \rho = \text{equation of state}; \quad c_s = \text{speed of sound}$
- Stability assumption: $0 < c_s < \sqrt{1/3}$
The Friedmann-Lemaître-Robertson-Walker (FLRW) family on \((-\infty, \infty) \times \mathbb{T}^3\)

Background solution ansatz:

\[
\tilde{p} = \tilde{p}(t), \quad \tilde{u}^\mu = (1, 0, 0, 0),
\]

\[
\tilde{g} = -dt^2 + a^2(t) \sum_{i=1}^{3} (dx^i)^2
\]

\[
\tilde{p}a^{3(1+c_s^2)} \equiv \bar{p} = \text{const}
\]

\[
\dot{a} = a \sqrt{\frac{\Lambda}{3}} + \frac{\text{const}}{3a^{3(1+c_s^2)}}, \quad a(0) = 1
\]

Friedmann’s equation

Can show: \(a(t) \sim e^{Ht}, \quad H = \sqrt{\frac{\Lambda}{3}} = \text{Hubble’s constant}\)
Accelerated expansion of the universe

S. Weinberg: “Almost all of modern cosmology is based on this Robertson-Walker metric.”

\[ \tilde{g} = -dt^2 + a^2(t) \sum_{i=1}^{3} (dx^i)^2 \]

\[ a(t) \sim e^{Ht} \]

**Notation:** \( \Omega(t) \overset{\text{def}}{=} \ln a(t) \)

\[ \Omega(t) \sim Ht \]

**Side remark:** \( \Lambda = 0 \implies a(t) \sim t^C, \quad C = \frac{2}{3(1+c_5^2)} \); \( t = 0 \sim \text{“Big Bang”} \)
S. Hawking: “We now have a good idea of [the universe’s] behavior at late times: the universe will continue to expand at an ever-increasing rate.”

Fundamental question

What happens if we slightly perturb the initial conditions of the special FLRW solution $\tilde{g}, \tilde{p}, \tilde{u}$?

Trustworthiness of FLRW solution $\iff$ its salient features are stable under small perturbations of the initial conditions

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The Cauchy problem for the Einstein equations

Fundamental facts:

- Local existence (Choquet-Bruhat, 1952)

\[ \text{Evolution equations} \]

\[ \text{Hyperbolic } \sim \text{ wave equations} \]

\[ \text{Constraint equations} \]

\[ \text{Elliptic} \]

\[ \text{Data:} \ 3\text{–manifold } \Sigma, \ \text{Riemannian metric } \hat{g}, \ \text{second fundamental form } \hat{K}, \ \text{matter data} \]

(Choquet-Bruhat & Geroch, 1969) Each data set launches a unique* “maximal solution:” the Maximal Globally Hyperbolic Development of the data (MGHD)
In other words \ldots

Rephrasing of fundamental question: What is the nature of the MGHD of data near that of the special FLRW solution $\tilde{g}, \tilde{p}, \tilde{u}$?
Continuation principle (to avoid blow-up)

Lemma (heuristic)

If a singularity forms at time $T_{\text{max}}$, then

$$\lim_{T \to T_{\text{max}}} \sup_{0 \leq t \leq T} S(t) = \infty$$

$S(t) = \text{suitably-defined Sobolev-type (i.e., } L^2_x - \text{type) norm}$

Moral conclusion: To avoid blow-up, show that $S(t)$ remains finite.
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Fundamental result with moral implications

Theorem (Littman, 1963)

Assume that

\[-\partial_t^2 \phi + \sum_{i=1}^{m} \partial_i^2 \phi = 0\]

in \( m \geq 2 \) spatial dimensions.

If and only if \( p = 2 \), we have

\[\|\nabla \phi(t)\|_{L^p_x} \leq f(t)\|\nabla \phi(0)\|_{L^p_x},\]

\[f = \text{independent of } \phi\]

Moral conclusion: \( L^2_x \)-based estimates (energy estimates) play an indispensable role in hyperbolic PDE.
Theorem (Littman, 1963)

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Norms \( 0 < q \ll 1, \ h_{jk} = e^{-2\Omega}g_{jk}, \ P = e^{3(1+c_s^2)\Omega}p, \ \Omega \sim Ht, \ H \overset{\text{def}}{=} \sqrt{\Lambda/3} \)
The FLRW stability theorem


Assume that $N \geq 3$ and $0 < c_s^2 < 1/3$. Then there exist constants $\epsilon_0 > 0$ and $C > 1$ such that for all $\epsilon \leq \epsilon_0$, if $S_N(0) \leq C^{-1}\epsilon$, then there exists a future global causal geodesically complete solution to the Euler-Einstein system with $\Lambda > 0$. Furthermore,

$$S_N(t) \leq \epsilon \quad (\Leftarrow \text{continuation principle})$$

holds for all $t \geq 0$. (i.e., the MGH is singularity-free in the future direction)

• Previous work: Anderson, Brauer/Rendall/Reula, Friedrich, Ringström, Isenberg, …
Strategy of proof: derive norm inequalities

Lemma (J.S. 2010)

\[ U_{N-1}^2(t) \leq U_{N-1}^2(t_1) + \int_{\tau=t_1}^{t} 2(3c_s^2 - 1 + q) e^{(1+q)\Omega} U_{N-1}^2 + CS_N U_{N-1} \, d\tau, \]

\[ S_{P-\bar{\rho},u^*;N}(t) \leq S_{P-\bar{\rho},u^*;N}(t_1) + C \int_{t_1}^{t} e^{-qH\tau} S_N^2 \, d\tau, \]

\[ S_{g_{00}+1;N}(t) \leq S_{g_{00}+1;N}(t_1) + \int_{t_1}^{t} -4qHS_{g_{00}+1;N} + Ce^{-qH\tau} S_N S_{g_{00}+1;N} \, d\tau, \]

\[ S_{g_0^*;N}(t) \leq S_{g_0^*;N}(t_1) + \int_{t_1}^{t} -4qHS_{g_0^*;N} + CS_{g_0^*;N} S_{g_0^*;N} + Ce^{-qH\tau} S_N S_{g_0^*;N} \, d\tau, \]

\[ S_{h^*;N}(t) \leq S_{h^*;N}(t_1) + \int_{t_1}^{t} He^{-qH\tau} S_{h^*;N} + Ce^{-qH\tau} S_N S_{h^*;N} \, d\tau, \]

\[ S_N \overset{\text{def}}{=} U_{N-1} + S_{P-\bar{\rho},u^*;N} + S_{g_{00}+1;N} + S_{g_0^*;N} + S_{h^*;N} \]

\[ H \overset{\text{def}}{=} \sqrt{\Lambda/3} \]
Geometric energy methods

Goal: geo-analytic structures
↔ useful coercive quantities

Ex: $\mathcal{L} = -\frac{1}{2}(g^{-1})^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi$

$\mathcal{L}'$s Euler-Lagrange eqn: $(g^{-1})^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = \mathcal{I}$

$$T^{\mu\nu} \overset{\text{def}}{=} 2 \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} + (g^{-1})_{\mu\nu} \mathcal{L}$$

$$= (g^{-1})^{\mu\alpha} (g^{-1})^{\nu\beta} \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} (g^{-1})^{\mu\nu} (g^{-1})^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi$$

Lemma

encodes the “almost conservation laws”

$$\nabla_\alpha T^{\alpha\mu} = -\mathcal{I}(g^{-1})^{\mu\alpha} \nabla_\alpha \phi$$
Vectorfield method of (John), Klainerman, Christodoulou

- $X^\nu = \text{“multiplier vectorfield”}$
- $(X)J^\mu [\nabla \phi, \nabla \phi] = (X)J^\mu = T^\mu_\nu X^\nu = \text{“compatible current”}$

\[
\nabla_\mu (X)J^\mu = -\mathcal{I}X^\alpha \nabla_\alpha \phi + T^\mu_\nu \nabla_\mu X^\nu
\]
The divergence theorem

Integral identity - one for each $X$!

\[
\int_{\Sigma_t} (X) J^\alpha \xi_\alpha = \int_{\Sigma_0} (X) J^\alpha \xi_\alpha + \int_\Omega -I X^\alpha \nabla_\alpha \phi + T^\mu_\nu \nabla_\mu X^\nu,
\]

$\xi = \text{oriented unit normal to } \Sigma$
Coerciveness example for $g = \text{diag}(-1, 1, 1, 1)$

- $\Sigma_t = \{(\tau, x) \mid \tau = t\}$
- $\xi_\mu = (-1, 0, 0, 0)$
- $X = (r - t)^2 \mathcal{L} + (r + t)^2 \mathcal{L}$, conformal Killing (Morawetz)
- $\mathcal{L} = \partial_t - \partial_r, \mathcal{L} = \partial_t + \partial_r$

$$(X) J^\mu \xi_\mu = T^\mu_\nu \xi_\mu X^\nu = \frac{1}{2} \left\{ (t - r)^2 (\nabla_{\mathcal{L}} \phi)^2 + (t + r)^2 (\nabla_{\mathcal{L}} \phi)^2 + (t^2 + r^2) | \nabla \phi |^2 \right\}$$

positivity $\sim$ dominant energy condition

Morawetz-type coercive energy estimate (with $|\phi|^2$ “correction”)

$$\int_{\Sigma_t} \left\{ (t - r)^2 |\nabla_{\mathcal{L}} \phi|^2 + (t + r)^2 |\nabla_{\mathcal{L}} \phi|^2 + (t^2 + r^2) | \nabla \phi |^2 + |\phi|^2 \right\} \, dx$$

$\lesssim \int_{\Sigma_0} \left\{ r^2 |\nabla_{\mathcal{L}} \phi|^2 + r^2 |\nabla_{\mathcal{L}} \phi|^2 + r^2 | \nabla \phi |^2 + |\phi|^2 \right\} \, dx$  

“Data”

$$+ \left| \int_\Omega \left\{ (r - t)^2 \nabla_{\mathcal{L}} \phi + (r + t)^2 \nabla_{\mathcal{L}} \phi \right\} \, dx \, dt \right|$$
**Null condition**

\[ L = \partial_t - \partial_r \]
\[ L = \partial_t + \partial_r \]

Angular directions

\[ t - r = \text{const} \]

\[ \boldsymbol{g} = \text{diag}(-1, 1, 1, 1) \]
\[ g_{\alpha\beta} L^\alpha L^\beta = 0 = g_{\alpha\beta} L_\alpha L_\beta \]
\[ g_{\alpha\beta} L^\alpha L^\beta = -2 \]

\[ (g^{-1})^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi = -\nabla_L \phi \nabla_L \phi + |\nabla \phi|^2 \]

\[ \sim \text{“null condition”} \]
Geometric methods: recent applications

- Proofs of local well-posedness: Wong, Smulevici, J.S.
- Global existence and almost global existence for nonlinear wave and wave-like equations: Katayama, Klainerman, Lindblad, Morawetz, Sideris, Metcalfe/Nakamura/Sogge, Keel/Smith/Sogge, Chae/Huh, Sterbenz, Yang, J.S.
- The analysis of singular limits in hyperbolic PDEs: J.S., J.S./Strain
- Geometric continuation principles: Klainerman/Rodnianski, Kommemii, Shao, Wang
- Wave maps: Ringström
Geometric methods: recent applications

- The decay of waves on black hole backgrounds & the stability of the Kerr black hole family: Dafermos/Rodnianski, Baskin, Blue, Andersson/Blue, Soffer/Blue, Holzegel, Schlue, Aretakis, Finster/Kamran/Smoller/Yau, Luk, Tataru/Tohaneanu
- Black hole uniqueness (in smooth class): Klainerman, Ionescu, Chrusciel, Alexakis, Nguyen, Wong
- The evolutionary formation of trapped surfaces for the vacuum Einstein equations: Christodoulou, Klainerman/Rodnianski, Yu
- The formation of shocks for the 3–dimensional Euler equations: Christodoulou
- The proof of Strichartz-type inequalities for quasilinear wave equations: Klainerman
Regular hyperbolicity à la Christodoulou

\[ \phi = (\phi^1, \ldots, \phi^n) : \mathbb{R}^m \to \mathbb{R}^n \]

\[ h_{\alpha\beta}^A \partial_\alpha \partial_\beta \phi^B = I_A \]

assume: \[ h_{\alpha\beta}^A = h_{\beta\alpha}^B \]

assume: “hyperbolicity”

Theorem (J.S. & W. Wong, in preparation)

Under the above conditions, one can completely identify all of the \( \xi_\mu, X^\nu \) that lead to a coercive (in the integrated sense) integral identity. Counterexamples if \( h_{\alpha\beta}^A \neq h_{\beta\alpha}^B \).

- Canonical stress:
  \[ Q^{\mu}_\nu \overset{\text{def}}{=} h^{(\mu\lambda)}_{AB} \partial_\lambda \phi^B \partial_\nu \phi^A - \frac{1}{2} \delta^{\mu}_\nu h^{(\kappa\lambda)}_{AB} \partial_\kappa \phi^A \partial_\lambda \phi^B \sim T^{\mu}_\nu \]
Geometry of the equations

\[ C_p^* \overset{\text{def}}{=} \text{Characteristic subset of } T^*_p \mathbb{R}^m \]
\[ \overset{\text{def}}{=} \{ \zeta \mid \det(h_{\alpha\beta}^A \zeta^\alpha \zeta^\beta) = 0 \} \]

\[ C_p \overset{\text{def}}{=} \text{Characteristic subset of } T_p \mathbb{R}^m; \text{ is the “dual”} \]
The Einstein-nonlinear electromagnetic equations

\[ \mathcal{M} = \mathbb{R}^{1+3} \]

\[
\begin{align*}
Ric_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= T_{\mu\nu} \\
(d\mathcal{F})_{\lambda\mu\nu} &= 0 \\
(d\mathcal{M})_{\lambda\mu\nu} &= 0
\end{align*}
\]

- In Maxwell’s theory:
  - \( \mathcal{M}_{\mu\nu}^{(\text{Maxwell})} = \ast \mathcal{F}_{\mu\nu} \) “null condition”
  - \( T_{\mu\nu}^{(\text{Maxwell})} = \mathcal{F}_\mu^\kappa \mathcal{F}_{\nu\kappa} - \frac{1}{4}g_{\mu\nu} \mathcal{F}_\kappa^\lambda \mathcal{F}^{\kappa\lambda} \)

- For many other Lagrangian theories:
  - \( \mathcal{M}_{\mu\nu} = \mathcal{M}_{\mu\nu}^{(\text{Maxwell})} + \) quadratic “error” terms
  - \( T_{\mu\nu} = T_{\mu\nu}^{(\text{Maxwell})} + \) cubic “error” terms
The stability of Minkowski spacetime

**Theorem (J.S. 2010; upcoming PNAS article)**

The Minkowski spacetime solution $\tilde{g} = \text{diag}(-1, 1, 1, 1)$, $\tilde{F} = 0$ is globally stable. The result holds for a large family of electromagnetic models that reduce to Maxwell’s theory in the weak-field limit.

- Proof: vectorfield method + regular hyperbolicity
- Stability: Christodoulou-Klainerman, Lindblad-Rodnianski, Bieri, Zipser, Choquet-Bruhat/Chrusciel/Loizelet
- Electromagnetism: Bialynicki-Birula, Boillat, Kiessling, Tahvildar-Zadeh, Carley, ... study e.g. the Maxwell-Born-Infeld model ($\leftrightarrow$ minimal surface equation)
- Above results $\sim$ the alternate electromagnetic models are physically reasonable
Future research directions

- MGHD - AIM workshop on shocks (with S. Klainerman, G. Holzegel, W. Wong, P. Yu)
- Sub-exponential expansion rates
  - Wave equations on expanding Lorentzian manifolds
- Alternative matter model: the relativistic Boltzmann equation (with R. Strain)
Future research directions continued

- Singularity formation in general relativity
- Regular hyperbolicity (with W. Wong)
  - classification of integral identities for nonlinear electromagnetism
Great open problem

- Maximal Globally Hyperbolic Developments
  - Weak Cosmic Censorship Hypothesis
    (R. Penrose, 1969)
    “Generic singularities are hidden inside black holes”

Figure: Schwarzschild solution of mass $M$ (M. Dafermos)
Wanted list

- Wanted: work on solving the Einstein constraint equations for some of the matter models:
  - Euler-Einstein with $\Lambda > 0$
  - Einstein-nonlinear electromagnetic
  - Einstein-Boltzmann

- Wanted: numerical evidence for the cases $c_s^2 \geq 1/3$

- Wanted: blow-up results away from the small-data regime
Thank you
The “Divergence Problem”

- Electric field surrounding an electron: $\mathbf{E}_{(Maxwell)} = -\frac{\hat{r}}{|r|^2}$

- Electrostatic energy: $e_{(Maxwell)} = \int_{\mathbb{R}^3} |\mathbf{E}_{(Maxwell)}|^2 d\text{Vol}$
  
  $= 4\pi \int_{r=0}^{\infty} \frac{1}{r^2} dr = \infty$

- Lorentz force: $\mathbf{F}_{(Lorentz)} = q\mathbf{E}_{(Maxwell)} = \text{Undefined}$
  (at the electron’s location)

- Analogous difficulties appear in QED

M. Born: “The attempts to combine Maxwell’s equations with the quantum theory (...) have not succeeded. One can see that the failure does not lie on the side of the quantum theory, but on the side of the field equations, which do not account for the existence of a radius of the electron (or its finite energy=mass).”
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