MIDTERM 2 - 18.01 - FALL 2014.

Name: ____________________________

Email: ____________________________

Please put a check by your recitation section.

<table>
<thead>
<tr>
<th>Instructor</th>
<th>Time</th>
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<tbody>
<tr>
<td>B. Yang</td>
<td>MW 10</td>
</tr>
<tr>
<td>M. Hoyois</td>
<td>MW 11</td>
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<tr>
<td>M. Hoyois</td>
<td>MW 12</td>
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<tr>
<td>X. Sun</td>
<td>MW 1</td>
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<tr>
<td>R. Chang</td>
<td>MW 2</td>
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<table>
<thead>
<tr>
<th>Problem #</th>
<th>Max points possible</th>
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<td>Total</td>
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Directions:

- Write your answers directly on the exam.
- No books, notes, or electronic devices can be used on the exam.
- Partial credit can be given if you show your work.
- Don’t forget to write your name and email and to indicate your recitation instructor above.

Good luck!
Problem 1. (15 points)
Find the linear approximation of the function $f(x) = \ln(x^2)$ near $x = 1$.

Solution: We compute that

$$f(x) = x \ln x, \quad f'(x) = \ln x + 1$$

$$\implies f(1) = 0, \quad f'(1) = 1.$$ 

Hence,

$$f(x) = x \ln x = f(1) + f'(1)(x - 1) + O((x - 1)^2)$$

$$= x - 1 + O((x - 1)^2).$$
Problem 2. (5 + 5 + 5 = 15 points)
a) State the mean value theorem. Furthermore, draw a picture of a function $f(x)$ defined on the interval $0 \leq x \leq 1$ that illustrates the mean value theorem. Your picture must include the secant line through $(0, f(0))$ and $(1, f(1))$, and you must explain what the mean value theorem says about this secant line.

b) Show that

$$
\sin x \leq x
$$

for all real numbers $x \geq 0$.

c) Show that

$$
\cos x \geq 1 - \frac{1}{2}x^2
$$

for all real numbers $x$.

Solution: a) The mean value theorem states that if $f$ is differentiable for $a < x < b$ and if $f$ is continuous for $a \leq x \leq b$, then there exists a point $c$ with $a < c < b$ such that

$$
\frac{f(b) - f(a)}{b - a} = f'(c).
$$
b) We set $f_1(x) = x - \sin x$. Then $f_1(0) = 0$ and $f_1'(x) = 1 - \cos x \geq 0$. Hence, $f_1$ is a non-decreasing function for all $x \geq 0$. Thus, $f_1(x) \geq f_1(0) = 0$ for all $x \geq 0$, that is, $x \geq \sin x$ as desired.

c) Since both sides of the inequality are even, we only need to prove the inequality for $x \geq 0$. To this end, we set $f_2(x) = \cos x - (1 - x^2/2)$. We next note that $f_2(0) = 0$, and by part b), $f_2'(x) = x - \sin x = f_1(x) \geq 0$ whenever $x \geq 0$. Thus, $f_2$ is a non-decreasing function for $x \geq 0$ and hence $f_2(x) \geq f_2(0) = 0$ whenever $x \geq 0$. Thus, $\cos x \geq 1 - x^2/2$ as desired.
Problem 3. (10 + 10 = 20 points) Compute the following two antiderivatives:

\[ a) \int \frac{\ln x}{x} \, dx \]
\[ b) \int (\cos x)^{1801} \sin x \, dx \]

Solution: a) We first set \( u = \ln x, \, du = \frac{1}{x} \, dx \). We then compute that
\[
\int \frac{\ln x}{x} \, dx = \int u \, du \\
= \frac{1}{2} u^2 + c \\
= \frac{1}{2} (\ln(x))^2 + c.
\]

b) We set \( u = \cos x, \, du = -\sin x \, dx \), and compute that
\[
\int (\cos x)^{1801} \sin x \, dx \\
= -\int u^{1801} \, du \\
= -\frac{1}{1802} u^{1802} + c \\
= -\frac{1}{1802} (\cos x)^{1802} + c.
\]
Problem 4. (15 points) A rectangle with sides parallel to the $x$ and $y$ axes lies inside the curve $x^4 + y^4 = 1$ in the $(x, y)$ plane. Note that the curve looks like a distorted circle. The bottom edge of the rectangle lies on the $x$ axis and its upper two vertices lie on the curve. Find the dimensions of the rectangle that maximize its area. To receive full credit, explain your reasoning and provide a justification that you have found the dimensions that lead to the maximal area.

Solution:

All candidate rectangles have vertices of the form $(-x,0), (x,0), (-x,(1-x^4)^{1/4}), (x,(1-x^4)^{1/4})$, where $0 \leq x \leq 1$. The area $A(x)$ of the candidate rectangle is

$$A(x) = \text{base} \times \text{height} = 2x(1-x^4)^{1/4}.$$ 

Since $A(0) = A(1) = 0$, the maximum does not occur at either of the endpoints. To find the critical points, we compute that

$$A'(x) = 2(1-x^4)^{1/4} - 2x^4(1-x^4)^{-3/4} = \frac{1-2x^4}{\sqrt{1-x^4}}.$$ 

Setting $A'(x) = 0$ to find the critical points $x_{\text{crit}}$, we find that

$$x_{\text{crit}} = \sqrt[4]{\frac{1}{2}}.$$ 

The corresponding height $y_{\text{crit}}$ on the curve is obtained by solving for $y$ when $x = x_{\text{crit}}$:

$$y_{\text{crit}} = \sqrt[4]{1-x_{\text{crit}}^4} = \sqrt[4]{\frac{1}{2}}.$$ 

Thus, the rectangle of maximal area has base width $2x_{\text{crit}} = 2\sqrt[4]{1/2}$ and height $y_{\text{crit}} = \sqrt[4]{1/2}$. 
Problem 5. (15 points) A solid cylinder of radius $r = 3$ meters and height $h = 4$ meters starts changing in size at time $t = 0$. Its radius begins shrinking at the rate of 1 meter per minute and its height begins growing at the rate of 2 meters per minute. At time $t = 0$, find the rate of change of the volume of the cylinder and decide whether the volume is increasing or decreasing.

Solution:
The volume of the cylinder is

$$V = \pi r^2 h.$$ 

Differentiating this relationship with respect to $t$ and applying the chain and product rules, we find that

$$\frac{dV}{dt} = 2\pi rh \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}. $$

Setting $r = 3$, $h = 4$, $\frac{dr}{dt} = -1$, and $\frac{dh}{dt} = 2$, we compute that

$$\frac{dV}{dt} = 24\pi \frac{dr}{dt} + 9\pi \frac{dh}{dt} = -24\pi + 18\pi = -6\pi \text{ meters}^3 \text{ per minute}. $$

Thus, the cylinder’s volume is decreasing.
Problem 6. (20 points) Sketch the graph of the function

\[ f(x) = \ln(x^2 + 1) - \ln 2. \]

Label the zeros of \( f(x) \) by “Z,” the critical points by “C,” and the inflection points by “I.” To receive full credit, you must clearly indicate:

i) Any discontinuities or points of non-differentiability \( f(x) \) may have.

ii) The limiting behavior of \( f(x) \) as \( x \to \pm \infty \).

iii) The regions on which \( f(x) \) is positive and the regions on which \( f(x) \) is negative.

iv) The regions on which \( f(x) \) is increasing and the regions on which \( f(x) \) is decreasing.

v) The regions on which \( f(x) \) is concave up and the regions on which \( f(x) \) is concave down.

Please be sure that you graph the correct function. If you accidentally graph a function other than the function \( f(x) \) written above, then we can only award a small amount of credit at most.

Solution: The graph of \( f(x) \) is given in the figure below.
To justify the graph shown, we first note $f$ has no discontinuities and that $\lim_{x \to \pm \infty} f(x) = \infty$.

We next note that setting $f(x) = 0$ leads to the equation $x^2 + 1 = 2$, which has the two solutions $x = \pm 1$.

We now compute the first and second derivatives of $f(x)$:

\[
f'(x) = \frac{2x}{x^2 + 1},
\]

\[
f''(x) = 2 \frac{1 - x^2}{(x^2 + 1)^2}.
\]

The above formulas imply that $x = 0$ is the only critical point (i.e., point where $f' = 0$) and $x = \pm 1$ are the two inflection points (i.e., points where $f'' = 0$). To indicate some of the other features of $f(x)$, we make the following table:
<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$f'(x)$</th>
<th>$f''(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; -1$</td>
<td>$+$</td>
<td>$-$ (decreasing)</td>
<td>$-$ (concave down)</td>
</tr>
<tr>
<td>$-1 &lt; x &lt; 0$</td>
<td>$-$</td>
<td>$-$ (decreasing)</td>
<td>$+$ (concave up)</td>
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<tr>
<td>$0 &lt; x &lt; 1$</td>
<td>$-$</td>
<td>$+$ (increasing)</td>
<td>$+$ (concave up)</td>
</tr>
<tr>
<td>$1 &lt; x$</td>
<td>$+$</td>
<td>$+$ (increasing)</td>
<td>$-$ (concave down)</td>
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