MIDTERM 1 - 18.01 - FALL 2014.

Name: ____________________________________________

Email: ____________________________________________

Please put a check by your recitation section.

<table>
<thead>
<tr>
<th>Instructor</th>
<th>Time</th>
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<tbody>
<tr>
<td>B. Yang</td>
<td>MW 10</td>
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<tr>
<td>M. Hoyois</td>
<td>MW 11</td>
</tr>
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<td>M. Hoyois</td>
<td>MW 12</td>
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<tr>
<td>X. Sun</td>
<td>MW 1</td>
</tr>
<tr>
<td>R. Chang</td>
<td>MW 2</td>
</tr>
</tbody>
</table>

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<tr>
<th>Problem #</th>
<th>Max points possible</th>
<th>Actual score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
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Directions:
- Write your answers directly on the exam.
- No books, notes, or electronic devices can be used on the exam.
- Partial credit can be given if you show your work.
- Don’t forget to write your name and email and to indicate your recitation instructor above.

Good luck!
Problem 1. (10 + 5 + 5 = 20 points)

a) Compute the derivative of the function \( f(x) = x^{x^2} \).

b) Compute the first and second derivatives of the function \( f(x) = xe^x \).

c) Let \( n \) be a positive integer. Based on your answer to part b), guess a formula for the \( n^{th} \) derivative of the function \( f(x) = xe^x \) and use mathematical induction to show that your formula is correct.

Solution:

1) 
\[
\begin{align*}
y &= x^{x^2} \\
\ln y &= x^2 \ln x \\
\frac{y'}{y} &= 2x \ln x + x \\
y' &= x^{x^2} (2x \ln x + x)
\end{align*}
\]

b) 
\[
\begin{align*}
y &= xe^x \\
y' &= xe^x + e^x \\
y'' &= xe^x + 2e^x.
\end{align*}
\]

c) We will use induction to show that \( \frac{d^n}{dx^n}(xe^x) = xe^x + ne^x \). The base case \( n = 1 \) was verified in part b). Suppose that \( \frac{d^n}{dx^n}(xe^x) = xe^x + ne^x \). Then

\[
\begin{align*}
\frac{d^{n+1}}{dx^{n+1}}(xe^x) &= \frac{d}{dx} \left\{ \frac{d^n}{dx^n}(xe^x) \right\} \\
&= \frac{d}{dx}(xe^x + ne^x) \\
&= xe^x + e^x + ne^x \\
&= xe^x + (n + 1)e^x.
\end{align*}
\]

We have thus proved that case \( n + 1 \) is a consequence of case \( n \), which completes the induction.
Problem 2. (5 + 10 = 15 points)

a) Let $f(x)$ be a function. State the analytic definition of $f'(x)$ (in terms of a limit).

b) Consider the function $f(x) = x|\cdot|$. Decide whether or not $f(x)$ is differentiable at the point $x_0 = 0$. To receive credit, your argument must involve the analytic definition of the derivative, and you must fully explain your reasoning.

Solution: a) 

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$ 

Solution: b) Since $f(0) = 0$,

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0} |\Delta x| = 0.$$ 

Thus, $f$ is differentiable at $x_0 = 0$ and $f'(0) = 0$. 
**Problem 3.** (15 points) Find the equation of the tangent line at the point \((x_0, y_0) = (1, \frac{\pi}{2})\) to the following curve in the \(xy\) plane:

\[
\sin(xy) + x^2 y = 1 + \frac{\pi}{2}.
\]

**Solution:** Implicitly differentiating the equation with respect to \(x\), we find that

\[
y' \left( x \cos(xy) + x^2 \right) + y \cos(xy) + 2xy = 0.
\]

We now set \((x, y) = (1, \frac{\pi}{2})\) in the above equation to deduce that

\[
y'(1 \cdot 0 + 1) + \frac{\pi}{2} \cdot 0 + 2 \cdot 1 \cdot \frac{\pi}{2} = 0,
\]

\[
y' = -\pi.
\]

Since the equation of the tangent line at \((x_0, y_0)\) is \(y - y_0 = f'(x_0)(x - x_0)\), when \((x_0, y_0) = (1, \frac{\pi}{2})\), the line can be expressed as

\[
y - \frac{\pi}{2} = -\pi(x - 1).
\]
Problem 4. (10 + 10 = 20 points) Compute the following limits by recognizing that they are equal to \( f'(x_0) \), where you have to figure out what \( f \) and \( x_0 \) are. You may not use L'Hôpital's rule, if you know what that is.

a) \[ \lim_{\Delta x \to 0} \frac{(10 + 2\Delta x)^{100} - 10^{100}}{\Delta x} \]

b) \[ \lim_{\Delta x \to 0} \frac{\pi/2 + \Delta x)^2 \cos(\pi/2 + \Delta x)}{\Delta x} \]

Solution: a)

\[
\lim_{\Delta x \to 0} \frac{(10 + 2\Delta x)^{100} - 10^{100}}{\Delta x} = 2 \lim_{\Delta x \to 0} \frac{(10 + 2\Delta x)^{100} - 10^{100}}{2\Delta x} = 2 \left. \frac{d}{dx} x^{100} \right|_{x=10} = 2 \cdot 100x^{99} \big|_{x=10} = 2 \cdot 100 \cdot 10^{99} = 2 \cdot 10^{101}.
\]

b)

\[
\lim_{\Delta x \to 0} \frac{(\pi/2 + \Delta x)^2 \cos(\pi/2 + \Delta x)}{\Delta x} = \left. \frac{d}{dx} (x^2 \cos x) \right|_{x=\pi/2} = \left\{ 2x \cos x - x^2 \sin x \right\} \big|_{x=\pi/2} = -\left( \frac{\pi}{2} \right)^2.
\]
Problem 5. (15 points) Let \( y = f(x) = \sin x + x \). Let \( g(y) \) be the inverse function of \( f \), that is, \( g(y) = x \) when \( y = f(x) \). Find an expression for \( \frac{d}{dy}g(y) \) when \( y = f(x) \).

You are allowed to express your answer in terms of \( x \), that is, in the form \( \frac{d}{dy}g(y) = \) expression involving \( x \).

Hint: Do not try to find a formula for \( g(y) \); you won’t be able to do it.

Solution:

By the chain rule, we have

\[
\frac{d}{dx} g(y) = \frac{d}{dy} g(y) \frac{dy}{dx} = 1.
\]

Since \( \frac{dy}{dx} = \cos x + 1 \), we have

\[
\frac{d}{dy} g(y) = \frac{1}{\cos x + 1}.
\]
Problem 6. (15 points) Suppose that $f(x)$ is a continuous function and $f(0) = -1$. Suppose furthermore that $f(x)$ is differentiable at every $x$ value except $x = -3$ and $x = 2$. Finally, suppose that the graph of $f'(x)$ is as follows:

Sketch the graph of $f(x)$ on the blank graph below. Your picture should be qualitatively accurate, but it doesn’t have to be quantitatively perfect.
Figure 2. Draw your graph of $f(x)$ here

Solution:

Figure 3. Graph of $f(x)$