a. Numerical integration
   (a) Riemann sums \( \int_a^b f(x) \, dx \approx \sum_{i=1}^n y_i \Delta x \)
      (i) \( y_i = f(x_i) \)
      (ii) The \( x_i \) belong to the \( i^{th} \) subinterval (the precise definition of \( x_i \) depends on whether you are using right sums, left sums, upper sums, lower sums, etc.)
      (iii) \( \Delta x = \frac{b-a}{n} \)
   (b) Trapezoid rule \( \int_a^b f(x) \, dx \approx \Delta x \left( y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{y_n}{2} \right) \)
      (i) \( \Delta x = \frac{b-a}{n} \)
      (ii) \( x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \ldots, \quad x_n = a + n\Delta x = b \)
   (c) Simpson’s method (\( n \) must be even)
      (i) \( \int_a^b f(x) \, dx \approx \frac{\Delta x}{3} \left( y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-3} + 2y_{n-2} + 4y_{n-1} + y_n \right) \)
      (ii) \( \Delta x = \frac{b-a}{n} \)
      (iii) \( x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \ldots, \quad x_n = a + n\Delta x = b \)

b. Computing \( \int (\sin x)^n (\cos x)^m \, dx \)
   (a) If \( m \) is odd, let \( u = \sin x, \quad du = \cos x \, dx \), and substitute \( (\cos x)^2 = 1 - (\sin x)^2 \) to transform the integral into a \( u \) integral.
   (b) If \( n \) is odd, interchange the roles of \( \sin x \) and \( \cos x \) and proceed as above.
   (c) If \( m, n \) both even, make repeated use of the trig identities \( (\cos x)^2 = \frac{1}{2}[1 + \cos(2x)] \) and \( (\sin x)^2 = \frac{1}{2}[1 - \cos(2x)] \).
   (d) If \( m \) is even, let \( u = \sec x, \quad du = \sec x \tan x \, dx \) and substitute \( (\tan x)^2 = (\sec x)^2 - 1 \) to transform the integral into a \( u \) integral.
   (e) If \( n \) is even, let \( u = \tan x, \quad du = (\sec x)^2 \, dx \) and substitute \( (\sec x)^2 = 1 + (\tan x)^2 \) to transform the integral into a \( u \) integral.
   (f) If \( m \) is odd and \( n \) is odd, then we haven’t studied how to evaluate the integral.

Inverse trig substitution
Midterm 4 - Review Sheet

(a) Is useful for evaluating \( \int \sqrt{ax^2 + bx + c} \, dx \) because it gets rid of the square root \( (a, b, c \) are constants).

(b) To evaluate \( \int \frac{dx}{\sqrt{x^2 + 1}} \), let \( x = \tan u \), \( dx = (\sec u)^2 \, du \), and substitute \((\sec u)^2 + 1 = (\sec u)^2\).

(c) To evaluate \( \int \frac{dx}{\sqrt{x^2 - 1}} \), let \( x = \sec u \), \( dx = \sec u \tan u \, du \), and substitute \((\sec u)^2 - 1 = (\tan u)^2\).

(d) To evaluate \( \int \frac{dx}{\sqrt{1 - x^2}} \), let \( x = \sin u \), \( dx = \cos u \, dx \), and substitute \( 1 - (\sin u)^2 = (\cos u)^2 \).

(e) To evaluate e.g. \( \int \frac{dx}{\sqrt{x^2 + 2x + 2}} \), first complete the square: \( x^2 + 2x + 2 = (x + 1)^2 + 1 \). Then let \( v = x + 1 \), \( dv = dx \), and proceed as above.

(f) You can draw a suitable right triangle to help you express the final answer in terms of \( x \).

e. Partial fractions

(a) Is a strategy for evaluation \( \int \frac{P(x)}{Q(x)} \), where \( P, Q \) are polynomials and the degree of \( P \) is < the degree of \( Q \).

(b) You have to factor \( Q(x) \) to its fullest extent.

(c) If \( Q(x) = (x + a)(x + b) \), guess \( \frac{P(x)}{Q(x)} = \frac{A}{x + a} + \frac{B}{x + b} \) and solve for the constants \( A, B \) using e.g. the cover-up method. Then integrate the right-hand side using prior techniques.

(d) If \( Q(x) = (x + a)(x + b)^2 \), guess \( \frac{P(x)}{Q(x)} = \frac{A}{x + a} + \frac{B}{x + b} + \frac{C}{(x + b)^2} \) and solve for the constants \( A, B, C \) (the cover-up method does not work for \( B \).) Then integrate the right-hand side using prior techniques.

(e) If \( Q(x) = (x + a)^2(x + b)^3 \), guess \( \frac{P(x)}{Q(x)} = \frac{A}{x + a} + \frac{B}{(x + a)^2} + \frac{C}{x + b} + \frac{D}{(x + b)^2} + \frac{E}{(x + b)^3} \), etc.

(f) If \( Q(x) = (x + a)(x^2 + b) \), guess \( \frac{P(x)}{Q(x)} = \frac{A}{x + a} + \frac{B_0 + B_1 x}{x^2 + b} \) and solve for the constants \( A, B_0, B_1 \) (the cover-up method works only on \( A \).) Then integrate the right-hand side using prior techniques. A similar idea would allow you to treat other quadratic factors (with no real roots) in place of \( x^2 + b \) (you might have to complete the square first).

(g) If \( Q(x) = (x + a)^2(x^2 + b)^2 \), guess \( \frac{P(x)}{Q(x)} = \frac{A}{x + a} + \frac{B}{(x + a)^2} + \frac{C_0 + C_1 x}{x^2 + b} + \frac{D_0 + D_1 x}{(x^2 + b)^2} \), etc.

f. Integration by parts

(a) Is simply the product rule in reverse

(b) \( \int u \, dv = uv - \int v \, du \)