Problem 1.
\( a) \) Find all solutions \( y = f(x) \) to the differential equation \( y' = x^2 y \ln y \).
\( b) \) Find the solution to the initial value problem \( y' = x^2 y \ln y, \ y(0) = e \).

Problem 2. Find the area in between the curves \( y = x^2, \ y = 1 - x^2 \) and \( x = 1 \).

Problem 3. Consider the region \( R \) bounded by the lines \( x = 4, \ x = 9, \ y = 0 \) and the curve \( y = x^{3/2} \). The region \( R \) is revolved around the line \( x = 1 \) to generate a solid \( S \). Find the volume of \( S \).

Problem 4. Let \( f(x) \) be a function defined for all \( x \geq 0 \). Suppose that \( \int_0^\infty f(x) \, dx \) is a finite number, where \( \int_0^\infty f(x) \, dx = \lim_{N \to \infty} \int_0^N f(x) \, dx \). For each \( N > 0 \), consider the function \( g_N(x) = f(Nx) \). Show that the average value of \( g_N(x) \) on the interval \([0, 2]\) converges to 0 as \( N \to \infty \).

Problem 5. Let
\[
F(x) = \int_0^1 e^{-t^2 x} \, dt.
\]
Show that when \( x > 0 \), we have
\[
F'(x) = -\frac{1}{2} x^{-1} F(x) + \frac{1}{2} x^{-1} e^{-x}.
\]

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Problem 7. Let \( N \geq 0 \) be an integer. Use a Riemann sum argument to show that
\[
\lim_{n \to \infty} \sum_{i=1}^n i^N = \frac{1}{N+1}.
\]
*Hint: Relate the above sums to Riemann sums that converge to the integral as \( n \to \infty \).*
Problem 1. a) Find all solutions \( y = f(x) \) to the differential equation \( y' = x^2y \ln y \).

**Solution:** We first write the differential equation as

\[
\frac{dy}{y \ln y} = x^2 \, dx.
\]

We now integrate both sides and use the substitution \( u = \ln y, \, du = \frac{1}{y} \, dy \) to deduce that

\[
\ln|\ln y| = \frac{1}{3} x^3 + C.
\]

Exponentiating twice, we deduce that

\[
y = e^{e^{(1/3)x^3}},
\]

\[
c = \pm e^C.
\]

b) Find the solution to the initial value problem \( y' = x^2y \ln y, \, y(0) = e \).

**Solution:** We have to solve for \( c \). Setting \( x = 0, \, y = e \) in the above formula for \( y \) in terms of \( x \), we deduce that

\[
e = e^{e^{(1/3)0^3}} = e^c,
\]

which implies that \( c = 1 \). The solution is therefore \( y = e^{e^{(1/3)x^3}} \).

Problem 2. Find the area in between the curves \( y = x^2, \, y = 1 - x^2 \) and \( x = 1 \).

**Solution:** The graphs of \( y = x^2, \, y = 1 - x^2 \) intersect when \( x^2 = 1 - x^2 \), or equivalently, when \( 2x^2 = 1 \). This equation has the solutions

\[
x = \pm \sqrt{\frac{1}{2}}.
\]

The region of interest can be split into two regions: \(-\sqrt{\frac{1}{2}} \leq x \leq \sqrt{\frac{1}{2}} \) and \( \sqrt{\frac{1}{2}} \leq x \leq 1 \). In the first region, the top curve is \( y = 1 - x^2 \) and the bottom one is \( y = x^2 \). The area of this region is therefore

\[
\int_{-\sqrt{\frac{1}{2}}}^{\sqrt{\frac{1}{2}}} (1 - x^2) - x^2 \, dx = \left[ x - \frac{2}{3}x^3 \right]_{-\sqrt{\frac{1}{2}}}^{\sqrt{\frac{1}{2}}}
\]

\[
= \frac{4}{3\sqrt{2}}.
\]

In the second region, the top curve is \( y = x^2 \) and the bottom one is \( y = 1 - x^2 \). The area of this region is therefore

\[
\int_{\sqrt{\frac{1}{2}}}^{1} x^2 - (1 - x^2) \, dx = \left[ \frac{2}{3}x^3 - x \right]_{\sqrt{\frac{1}{2}}}^{1}
\]

\[
= \frac{1}{3} + \frac{2}{3\sqrt{2}}.
\]
Therefore, the total area of the two regions is
\[
\frac{4}{3\sqrt{2}} - \frac{1}{3} + \frac{2}{3\sqrt{2}} = \sqrt{2} - \frac{1}{3}.
\]

**Problem 3.** Consider the region \( R \) bounded by the lines \( x = 4, \ x = 9, \ y = 0 \) and the curve \( y = x^{3/2} \). The region \( R \) is revolved around the line \( x = 1 \) to generate a solid \( S \). Find the volume of \( S \).

**Solution:** We divide \( S \) into cylindrical shells whose axes are parallel to the \( y \)-axis. For \( 4 \leq x \leq 9 \), each shell has a height \( x^{3/2} \), a radius \( x - 1 \), and a width \( dx \). Thus, the volume of the shell is
\[
dV = 2\pi (x - 1) x^{3/2} dx.
\]
To find the volume \( V \) of \( S \), we integrate \( dV \):
\[
V = \int dV = \int_{x=4}^{x=9} 2\pi (x - 1) x^{3/2} dx
\]
\[
= 2\pi \left[ \frac{2}{7} x^{7/2} - \frac{2}{5} x^{5/2} \right]_{x=4}^{x=9}
\]
\[
= 2\pi \left( \frac{2}{7} 3^{7/2} - \frac{2}{5} 3^{5/2} + \frac{2}{7} 2^{7/2} - \frac{2}{5} 2^{5/2} \right).
\]

**Problem 4.** Let \( f(x) \) be a function defined for all \( x \geq 0 \). Suppose that \( \int_{0}^{\infty} f(x) \, dx \) is a finite number, where \( \int_{0}^{\infty} f(x) \, dx = \lim_{N \to \infty} \int_{0}^{N} f(x) \, dx \). For each \( N > 0 \), consider the function \( g_N(x) = f(Nx) \). Show that the average value of \( g_N(x) \) on the interval \([0,2]\) converges to 0 as \( N \to \infty \).

**Solution:** By definition, the average value of \( g_N(x) \) on \([0,2]\) is \( \frac{1}{2} \int_{0}^{2} f(Nx) \, dx \). By making the substitution \( u = Nx, \ du = N \, dx \), we see that the average value of \( g_N \) is
\[
\frac{1}{2N} \int_{u=0}^{u=2N} f(u) \, du.
\]
By assumption, we have that \( \lim_{N \to \infty} \int_{u=0}^{u=2N} f(u) \, du = I \), where \( I \) is some finite number. Therefore, it follows that
\[
\lim_{N \to \infty} \left\{ \frac{1}{2N} \int_{u=0}^{u=2N} f(u) \, du \right\}
\]
\[
= \lim_{N \to \infty} \frac{1}{2N} \times \lim_{N \to \infty} \int_{u=0}^{u=2N} f(u) \, du
\]
\[
= 0 \times I = 0
\]
as desired.

**Problem 5.** Let
\[
F(x) = \int_{0}^{1} e^{-t^2 x} \, dt.
\]
Show that when $x > 0$, we have

$$F'(x) = -\frac{1}{2}x^{-1}F(x) + \frac{1}{2}x^{-1}e^{-x}.$$ 

**Solution:** We make the change of variables $u = t\sqrt{x}$, $du = dt\sqrt{x}$. Then we have the identity

$$F(x) = \int_0^1 e^{-t^2x} dt = x^{-1/2} \int_0^{\sqrt{x}} e^{-u^2} du.$$

We now use the product rule, the second fundamental theorem of calculus, and the chain rule to deduce that

$$F'(x) = \frac{d}{dx} \left(x^{-1/2} \int_0^{\sqrt{x}} e^{-u^2} du\right)$$

$$= \left(\frac{d}{dx}x^{-1/2}\right) \int_0^{\sqrt{x}} e^{-u^2} du + x^{-1/2} \frac{d}{dx} \int_0^{\sqrt{x}} e^{-u^2} du$$

$$= -\frac{1}{2}x^{-3/2} \int_0^{\sqrt{x}} e^{-u^2} du + x^{-1/2}e^{-(\sqrt{x})^2} \frac{d}{dx}\sqrt{x}$$

$$= -\frac{1}{2}x^{-3/2} \int_0^{\sqrt{x}} e^{-u^2} du + \frac{1}{2}x^{-1}e^{-(\sqrt{x})^2}$$

$$= -\frac{1}{2}x^{-3/2} \int_0^{\sqrt{x}} e^{-u^2} du + \frac{1}{2}x^{-1}e^{-x}$$

$$= -\frac{1}{2}x^{-1}F(x) + \frac{1}{2}x^{-1}e^{-x}$$

as desired.

**Problem 6.** pg. 229 problem 16

**Solution:** We first calculate the area $A(r)$ of the square cross section of the solid that is located a perpendicular distance $r$ from the center of the wooden sphere (where $0 \leq r \leq a$ accounts for half of the solid). By the pythagorean theorem, the diagonal length of the square cross section is $d = 2\sqrt{a^2 - r^2}$. Since the area $A$ of a square with a diagonal of length $d$ is $A = d^2/2$, it follows that $A(r) = 2(a^2 - r^2)$. Therefore, the volume of a thin slice of width $dr$ is $A(r)dr$.

By symmetry, to find the total volume, we can double the volume of one half of the solid (which is obtained by integrating $A(r)dr$ from $r = 0$ to $r = a$):

$$V = 2 \int_{r=0}^a 2(a^2 - r^2) dr$$

$$= 4 \left[a^2r - \frac{1}{3}r^3\right]_{r=0}^{r=a}$$

$$= \frac{8}{3}a^3.$$
Problem 7. Let $N \geq 0$ be an integer. Use a Riemann sum argument to show that

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} i^N}{n^{N+1}} = \frac{1}{N+1}.$$

Hint: Relate the above sums to Riemann sums that converge to the integral as $n \to \infty$.

We first note that

$$\int_{0}^{1} x^N \, dx = \frac{1}{N+1}.$$

We then approximate the integral using right Riemann sums with $n$ equal intervals. More precisely, let $f(x) = x^N$. Then with $\Delta x = 1/n$, the right Riemann sum is

$$\sum_{i=1}^{n} f(i/n) \Delta x = \sum_{i=1}^{n} \left( \frac{i}{n} \right)^N \frac{1}{n} = \frac{\sum_{i=1}^{n} i^N}{n^{N+1}}.$$

As $n \to \infty$, the Riemann rectangles become infinitely thin and the Riemann sums converge to the integral $\frac{1}{N+1}$. That is,

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} i^N}{n^{N+1}} = \frac{1}{N+1},$$

as desired.