Problem 1. Compute the second derivative of the function \( f(x) = \arctan x \).

Problem 2. Compute the derivative of the function \( f(x) = \sin(x)x^x \).

Problem 3. Compute \( \lim_{x \to \pi/2} \frac{\cos(3x)}{\cos(x)} \). At this point in the course, you are forbidden from using L'Hôpital’s rule.

Problem 4. Compute the derivative \( \frac{dy}{dx} \) for the curve \( x^2 + y^3 + xy = 7 \) at the point \((x, y) = (2, 1)\).

Problem 5. Find the equation for the tangent line to \( y = \ln(x/3) \) at \( x = 3e \).

Problem 6. Consider the function

\[
 f(x) = \begin{cases} 
 -\frac{\sin x}{2x} & \text{if } x < 0, \\
 0 & \text{if } x = 0, \\
 \frac{\cos x - 1}{x^2} & \text{if } x > 0. 
\end{cases}
\]

Is \( f(x) \) continuous at \( x = 0 \)? If not, then is the discontinuity removable?

Problem 7. Prove the quotient rule \((u/v)' = (u'v - uv')/v^2\) using only the definition of a derivative.

Problem 8. Let \( r \) be a real number. Compute \( \lim_{h \to 0} \frac{(1+2h)^r - 1}{h} \) by interpreting this limit as a derivative.
Solutions

**Problem 1.** Compute the second derivative of the function \( f(x) = \arctan x \).

**Solution:**

\[
\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}, \quad \text{(you should know how to prove this)},
\]

\[
\frac{d^2}{dx^2} \arctan x = \frac{d}{dx} \frac{1}{1 + x^2} = \frac{-2x}{(1 + x^2)^2}.
\]

**Problem 2.** Compute the derivative of the function \( f(x) = \sin(x) x^x \).

**Solution:**

We first compute the derivative of \( x^x \) using logarithmic differentiation:

\[
y = x^x, \\
\ln y = \ln(x^x) = x \ln x, \\
\frac{y'}{y} = 1 + \ln x, \\
y' = (1 + \ln x)x^x.
\]

We now compute the derivative of \( x^x \):

\[
z = x^x, \\
\ln z = \ln(x^x) = x \ln x, \\
\frac{z'}{z} = \ln x \frac{d}{dx} x^x + x^x \frac{d}{dx} \ln x = \ln x (1 + \ln x) x^x + x^x \frac{1}{x}, \\
z' = x^x \left\{ \ln x (1 + \ln x) x^x + x^x \frac{1}{x} \right\}.
\]

Finally, we compute the derivative of \( \sin(x) x^x \) using the product rule:

\[
\frac{d}{dx} (\sin(x) x^x) = x^x \frac{d}{dx} \sin(x) + \sin(x) \frac{d}{dx} x^x \\
= x^x \cos x + \sin(x) x^x \left\{ \ln x (1 + \ln x) x^x + x^x \frac{1}{x} \right\}.
\]

**Problem 3.** Compute \( \lim_{x \to \pi/2} \frac{\cos(3x)}{\cos(x)} \). At this point in the course, you are forbidden from using L'Hôpital’s rule.

**Solution:**
\[ \lim_{x \to \pi/2} \frac{\cos(3x)}{\cos(x)} = \lim_{x \to \pi/2} \frac{\cos(3x) - \cos(3\pi/2)}{\cos(x) - \cos(\pi/2)} \]
\[ = 3 \lim_{x \to \pi/2} \frac{\cos(h) - \cos(3\pi/2)}{h - 3\pi/2} \times \frac{x - \pi/2}{\cos(x) - \cos(\pi/2)} \]
\[ = 3 \lim_{h \to 3\pi/2} \frac{\cos(h) - \cos(3\pi/2)}{h - 3\pi/2} \times \frac{1}{\lim_{h \to \pi/2} \frac{\cos(h) - \cos(\pi/2)}{h - \pi/2}} \]
\[ = 3 \frac{d}{du} \cos(u)|_{u=3\pi/2} \times \frac{1}{\frac{d}{du} \cos(u)|_{u=\pi/2}}, \]
\[ = 3(-\sin(3\pi/2)) \times \frac{1}{-\sin(\pi/2)} \]
\[ = 3(1)(-1) = -3. \]

**Problem 4.** Compute the derivative \( \frac{dy}{dx} \) for the curve \( x^2 + y^3 + xy = 7 \) at the point \((x, y) = (2, 1)\).

**Solution:**

\[ 2x + 3y^2y' + xy' + y = 0, \]
\[ y' = -\frac{2x + y}{3y^2 + x} = -\frac{5}{5} = -1. \]

**Problem 5.** Find the equation for the tangent line to \( y = f(x) = \ln(x/3) \) at \( x = 3e \).

**Solution:**

\[ f(3e) = \ln e = 1, \]
\[ f'(x) = \frac{1}{x/3} \times \frac{1}{3} = \frac{1}{x}, \]
\[ f'(3e) = \frac{1}{3e}. \]

Tangent line (with \( x_0 = 3e \)):

\[ y - f(x_0) = f'(x_0)(x - x_0), \]
\[ y - 1 = \frac{1}{3e}(x - 3e). \]

**Problem 6.** Consider the function

\[ f(x) = \begin{cases} \frac{-\sin x}{2x} & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ \frac{\cos x - 1}{x^2} & \text{if } x > 0. \end{cases} \]
Is \( f(x) \) continuous at \( x = 0 \)? If not, then is the discontinuity removable?

**Solution:**

We recall the following important limits, which have been previously investigated in this course:

\[
\lim_{x \to 0^-} -\frac{\sin x}{2x} = -\frac{1}{2}, \\
\lim_{x \to 0^+} \frac{\cos x - 1}{x^2} = -\frac{1}{2}.
\]

Therefore, since \( \lim_{x \to 0} f(x) = -\frac{1}{2} \neq f(0) \), \( f(x) \) is not continuous at \( x = 0 \).

However, if we redefine \( f \) by setting \( f(0) = -\frac{1}{2} \), then for the new \( f \), we have \( \lim_{x \to 0} f(x) = f(0) \). Thus, the redefined \( f \) is continuous (and hence the original \( f \) has a removable discontinuity at \( x = 0 \)).

**Problem 7.** Prove the quotient rule \((u/v)' = (u'v - uv')/v^2\) using only the definition of a derivative.

**Solution:**

Given a point \( x \) and a small number \( \Delta x \), we define

\[
\Delta u = u(x + \Delta x) - u(x), \quad \Delta v = v(x + \Delta x) - v(x).
\]

Since \( u, v \) are differentiable (and therefore continuous), we have

\[
\lim_{\Delta x \to 0} \Delta u = 0, \\
\lim_{\Delta x \to 0} \Delta v = 0, \\
\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = u', \\
\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = v'.
\]

Using the above facts, we have that

\[
\left( \frac{u}{v} \right)' = \lim_{\Delta x \to 0} \frac{\frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}}{\Delta x} = \lim_{\Delta x \to 0} \frac{(u + \Delta u)v - (v + \Delta v)u}{v(v + \Delta v)} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \frac{v}{v + \Delta v} - \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \frac{u}{v + \Delta v} = \frac{u'v}{v^2} - \frac{u'v}{v^2} + \frac{u'v - uv'}{v^2} = \frac{u'v - uv'}{v^2}.
\]
Problem 8. Let $r$ be a real number. Compute $\lim_{h \to 0} \frac{(1+2h)^r - 1}{h}$ by interpreting this limit as a derivative.

Solution:

$$\lim_{h \to 0} \frac{(1+2h)^r - 1}{h} = \frac{d}{dx} (1+2x)^r |_{x=0} = r(1+2x)^{r-1} \times 2 |_{x=0} = 2r.$$