a. Parametric curves
   (a) Are curves in the \((x, y)\) plane expressed as
   \[
   x = F(t), \\
   y = G(t),
   \]
   \[a \leq t \leq b,\] where \(t\) is called the parameter.

b. Arc length of a curve
   (a) Arc length is equal to \(\int_a^b ds\).
   (b) \(a\) is the parameter starting point, \(b\) is the parameter end point.
   (c) For curves in parametric form,
   \[
   ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{(F'(t))^2 + (G'(t))^2} dt
   \]
   (Pythagorean theorem).
   (d) For curves \(y = f(x)\), the formula reduces to
   \[
   ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + (f'(x))^2} dx
   \]
   (and \(x\) is the parameter).

c. Surface area of a solid formed by revolving a curve around the \(x\)-axis (for revolution around the \(y\)-axis, interchange the roles of \(x\) and \(y\) in everything that follows)
   (a) Divide the surface into small strips that are portions of cones (the cone strip radii are parallel to the \(y\)-axis, and the cone strip axes of symmetry are parallel to the \(x\)-axis).
   (b) Surface area is given by
   \[
   \int \text{ conical strip circumference } \times \text{ slant edge length} \\
   = \int 2\pi \text{ conical strip radius } \times ds
   \]
   \[
   = \int_{t=a}^{t=b} 2\pi G(t) \sqrt{\left(F'(t)\right)^2 + \left(G'(t)\right)^2} dt.
   \]
   (c) \(a\) is the parameter starting point, \(b\) is the parameter end point.
   (d) For curves \(y = f(x)\), the formula reduces to
   \[
   \int_{x=a}^{x=b} 2\pi f(x) \sqrt{1 + (f'(x))^2} dx.
   \]

d. Polar coordinates
   (a) \(x = r \cos \theta, \ y = r \sin \theta\)
   (b) In the standard formulation, \(r = \sqrt{x^2 + y^2}, \ \theta\) is the polar angle, and \(0 \leq \theta < 2\pi\)
(c) Area in polar coordinates: Area under the curve $r = f(\theta)$ in between the angles $\theta_1$ and $\theta_2$ is given by

$$\text{Area} = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta = \frac{1}{2} \int_{\theta_1}^{\theta_2} |f(\theta)|^2 \, d\theta$$

e. L'Hôpital’s rule

(a) Sometimes allows one to evaluate limits of the form $\lim_{x \to 0} \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are differentiable functions, $a$ is a finite number, $f(a) = g(a) = 0$, and $\lim_{x \to a} f'(x) = L$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = L$. Furthermore, it is sometimes true that $L = \frac{f'(a)}{g'(a)}$ (for example, when $f'(x)$ and $g'(x)$ are continuous at $x = a$ and $g'(a) \neq 0$).

(b) Many of the above limits can be massaged into the form $\lim_{x \to 0} \frac{f(x)}{g(x)}$ or $\lim_{x \to \infty} \frac{f(x)}{g(x)}$, where L'Hôpital’s rule can sometimes directly be applied. For example, the $0^0$ case can be massaged into the $0^0$ case with the help of $\ln$.

(c) In the “$0^0$” case: If $f, g$ are differentiable functions, $a$ is a finite number, $f(a) = g(a) = 0$, and $\lim_{x \to a} f'(x) = L$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = L$. Furthermore, it is sometimes true that $L = \frac{f'(a)}{g'(a)}$ (for example, when $f'(x)$ and $g'(x)$ are continuous at $x = a$ and $g'(a) \neq 0$).

(d) In the “$\infty$” case: If $f, g$ are differentiable functions, $a$ is a finite number, $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$, and $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

(e) Analogous statements hold if we replace $\lim_{x \to a}$ with $\lim_{x \to \infty}$ or $\lim_{x \to -\infty}$.

f. Improper integrals

(a) If $f(x)$ is continuous for $0 \leq x < \infty$, then by definition, $\int_0^\infty f(x) \, dx = \lim_{M \to \infty} \int_0^M f(x) \, dx$.

(i) If the limit exists, we say the improper integral converges. Otherwise, we say it diverges.

(b) If $f(x)$ is continuous for $0 < x \leq b$ but is not continuous at $x = a$, then by definition, $\int_a^b f(x) \, dx = \lim_{x_0 \to a^+} \int_{x_0}^b f(x) \, dx$.

(i) If the limit exists, we say the improper integral converges. Otherwise, we say it diverges.

g. Infinite series

(a) Are series of the form $\sum_{k=0}^\infty a_k = a_0 + a_1 + a_2 + a_3 + \cdots$

(b) By definition, $\sum_{k=0}^\infty a_k = \lim_{M \to \infty} S_M$, where $S_M = \sum_{k=0}^M a_k = a_0 + a_1 + a_2 + \cdots + a_M$ is the $M$th partial sum.

(i) If $\lim_{M \to \infty} S_M$ exists, we say the series converges. Otherwise, we say it diverges.

(c) Geometric series: $\sum_{k=0}^\infty x^k = \frac{1}{1-x}$ if $|x| < 1$. $\sum_{k=0}^\infty x^k$ diverges if $|x| \geq 1$.

(d) Comparison: If $0 \leq a_k \leq b_k$ for all large $k$, and if $\sum_{k=0}^\infty a_k$ diverges, then $\sum_{k=0}^\infty b_k$ diverges too (divergence of smaller $\implies$ divergence of bigger). If $0 \leq a_k \leq b_k$ for all large $k$, and if $\sum_{k=0}^\infty b_k$ converges, then $\sum_{k=0}^\infty a_k$ converges too (convergence of bigger $\implies$ convergence of smaller).

(e) Limit comparison test: If $a_k \geq 0, b_k \geq 0$ for all large $k$ and $a_k \sim b_k$, then $\sum_{k=0}^\infty a_k$ converges if and only if $\sum_{k=0}^\infty b_k$ converges. Here, $a_k \sim b_k$ means that there exists a non-zero number $L$ such that $\lim_{k \to \infty} \frac{a_k}{b_k} = L$.

(f) Integral comparison: If $f(x)$ is continuous, $f(x) \geq 0$ for all $x$, and $f(x)$ is decreasing for all large $x$, then $\sum_{k=0}^\infty f(k)$ converges if and only if the improper integral $\int_{x=0}^\infty f(x) \, dx$ converges.

h. Taylor’s series with base point $b = 0$
(a) For \( x \) near 0 : \( f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \)

(b) \( a_n = \frac{f^{(n)}}{n!}(0) \), where \( f^{(n)} \) is the \( n^{th} \) derivative of \( f \)

(c) \( e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \)

(d) \( \ln(1 + x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \pm \cdots \)

(e) \( \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \pm \cdots \)

(f) \( \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \pm \cdots \)

i. Taylor’s series with base point \( b \)

(a) For \( x \) near \( b \) : \( f(x) = a_0 + a_1 (x-b) + a_2 (x-b)^2 + a_3 (x-b)^3 + \cdots \)

(b) \( a_n = \frac{f^{(n)}(b)}{n!} \)

(c) \( x^A = b^A + Ab^{A-1} (x-b) + \frac{A(A-1)b^{A-2}}{2!} (x-b)^2 + \frac{A(A-1)(A-2)b^{A-3}}{3!} (x-b)^3 + \cdots \)