Problem 1 Recall that for a finite, connected graph $G = (V \cup \partial V, E)$ with $\partial V \neq \emptyset$, the discrete Gaussian free field (DGFF) $h$ on $G$ is the random function $h : V \cup \partial V \to \mathbb{R}$ with density

$$\frac{1}{\mathcal{Z}} \exp \left( -\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)$$

with respect to Lebesgue measure on $\mathbb{R}^{|V|}$ and constrained to satisfy $h|_{\partial V} \equiv 0$. The sum is over all pairs $\{x, y\} \in E$ and $\mathcal{Z}$ is a normalizing constant so that this defines a probability measure.

Suppose that $D \subseteq \mathbb{C}$ is bounded. Recall also that for $s \in \mathbb{R}$ we defined $(-\Delta)^s f = \sum_n \left(-\lambda_n\right)^s \alpha_n \phi_n$ for $f = \sum_n \alpha_n \phi_n$ where $(\lambda_n)$ are the negative eigenvalues of $\Delta$ and $(\phi_n)$ are the corresponding eigenvectors, normalized to be an orthonormal basis of $L^2(D)$. We then set $(f, g)_{H^s} = \left((-\Delta)^{s/2} f, (-\Delta)^{s/2} g\right)_{L^2}$ and let $H^s(D)$ be the closure of $C^\infty_0(D)$ under $(\cdot, \cdot)_{H^s}$.

For each $n$, let $h_n$ be a DGFF on the subgraph $G_n = (V_n \cup \partial V_n, E_n)$ of $\frac{1}{n} \mathbb{Z}^2$ which consists of those vertices in $\frac{1}{n} \mathbb{Z}^2$ which are contained in $[0, 1]^2$. The boundary vertices are those which are contained in $\partial [0, 1]^2$. For $f \in C^\infty_0([0, 1]^2)$, let

$$\xi_n(f) = \frac{1}{\sqrt{2\pi}} \sum_{b \in E_n} \nabla f(b) \nabla h_n(b)$$

where $\nabla g(b) = g(y) - g(x)$ for $b = (x, y) \in E_n$.

Show that $\xi_n$ converges to the continuum GFF on $[0, 1]^2$ (with zero boundary conditions) using the following steps.

1. Explain why, for $f \in C^\infty_0(D)$, we have that $\xi_n(f)$ is a Gaussian random variable with mean 0 and variance $(2\pi)^{-1} \sum_{b \in E_n} (\nabla f(b))^2$.

2. Explain why

$$\frac{1}{2\pi} \sum_{b \in E_n} (\nabla f(b))^2 \to \|f\|_{H^s}^2 = \frac{1}{2\pi} \int_{[0,1]^2} (\nabla f(x))^2 \, dx \quad \text{as} \quad n \to \infty.$$

3. Explain why $\psi_j = (-\lambda_j)^{-\frac{1}{2}} \phi_j$ is an orthonormal basis of $H^4([0, 2]^2)$.

4. Assume that there exists a constant $a_0 > 0$ such that $\sum_{b \in E_n} (\nabla \phi_j(b))^2 \leq a_0 j^2$ for all $n, j \in \mathbb{N}$. Show that there exists $a > 0$ such that

$$\sum_j \mathbb{P} \left[ \xi_n(\psi_j) \geq j^{-1/2-a} \right] < \infty$$

Date: December 7, 2014.
(Recall the Weyl asymptotics: \(-\lambda_j / j \to c_0\) as \(j \to \infty\) for a constant \(c_0 \in (0, \infty)\) depending only on \(D\). You may use without proof the fact that if \(Z \sim \mathcal{N}(0, 1)\) then
\[
P[Z \geq \lambda] \sim \sqrt{\frac{2}{\pi}} \lambda^{-1} e^{-\lambda^2 / 2} \quad \text{as} \quad \lambda \to \infty
\]
where \(f(\lambda) \sim g(\lambda)\) if and only if \(f(\lambda) / g(\lambda) \to 1\) as \(\lambda \to \infty\).

(5) Using the Borel-Cantelli lemma, show that there almost surely exists \(C < \infty\) (random)
\[
|\xi_n(\psi_j)| \leq \frac{C}{j^{1/2+\varepsilon}} \quad \text{for all} \quad j
\]

Explain why the law of \(C\) can be made to be tight in \(n\).

(6) Suppose that \(f \in H^4([0, 1]^2)\). Using the previous part, explain why there almost surely exists \(C < \infty\) (random but whose law is tight with \(n\)) such that
\[
|\xi_n(f)| \leq C\|f\|_{H^4([0, 1]^2)}
\]

Use this to conclude that \(\|\xi_n\|_{H^{-s}([0, 1]^2)} \leq C\).

(7) Explain why all of the parts together imply the result.

(8) Is it possible to prove convergence in \(H^{-s}([0, 1]^2)\) for some \(s < 4\) (i.e., is \(s = 4\) the best possible)?

**Problem 2** Using the notation of the previous problem, show that for each \(\delta > 0\) we have that
\[
P \left[ \max_{x \in V_n} h(x) \geq \left( \sqrt{\frac{2}{\pi}} + \delta \right) \log n \right] \to 0 \quad \text{as} \quad n \to \infty
\]

using the following steps. (Letting \(G_n(x, y)\) be given by the expected number of steps that a simple random walk starting from \(x\) spends at \(y\) before exiting \(V_n\), recall from class that \(\text{cov}(h(x), h(y)) = d^{-1}_y G_n(x, y)\) where \(d_y\) is the degree at \(y\).)

(1) Explain why \(G_n(x, y) = \sum_{t=0}^\infty p^t_n(x, y)\) where \(p^t_n(x, y)\) is the transition kernel for simple random walk on \(V_n\) stopped upon exiting \(V_n\). That is, \(p^t_n(x, y)\) is the probability that a simple random walk started from \(x\) hits \(y\) at time \(t\) before exiting \(V_n\).

(2) Let \(p^t\) be the transition kernel for simple random walk on \(\mathbb{Z}^2\). That is, \(p^t(x, y)\) is the probability that a simple random walk starting from \(x\) is at \(y\) at time \(t\). (This differs from \(p^t_n\) because \(p^t_n\) is for the walk stopped on exiting \(V_n\).) The local central limit theorem\(^1\) implies that there exists a constant \(c_0 > 0\) such that
\[
|p^n(x, y) + p^{n+1}(x, y) - 2p^n(x, y)| \leq \frac{c_0}{n^2}
\]
where
\[
p^n(x, y) = \frac{1}{\pi n} e^{-||x-y||^2/n}.
\]

Using the local central limit theorem and the Markov property for simple random walk, explain why there exists a constant \(c_1 > 0\) such that
\[
p^{\alpha n^2}_n(x, x) \leq e^{-c_1 n} \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad \alpha > 0.
\]

\(^1\)For a reference, see [1, Theorem 2.1.3]; the actual local central limit theorem is significantly stronger than the result stated here.
Problem 3 Suppose that $f$ is a distribution on $D$ with $\Delta f = 0$ in the distributional sense. That is, if $g \in C_0^\infty(D)$, then $(f, \Delta g) = 0$. Show that $f \in C^\infty(D)$ and $\Delta f = 0$ using the following steps.

1. Let $\phi$ be a radially symmetric $C_0^\infty$ bump function supported in $D$. In other words, $\phi(x) \geq 0$ for all $x$, $\phi(x)$ depends only on $|x|$, $\phi(x) = 0$ for $|x| \geq 1$, and $\int \phi = 1$. For each $\epsilon > 0$, let

$$f_\epsilon(x) = \epsilon^{-2} \int f(y) \phi \left( \frac{x - y}{\epsilon} \right) \, dy.$$ 

Explain why $f_\epsilon$ is $C^\infty$ in $D_\epsilon = \{ z \in D : \text{dist}(z, \partial D) \geq \epsilon \}$.

2. Fix $\delta > 0$ and let $x \in D_\delta$. Explain why $f_\epsilon(x)$ does not depend on the value of $\epsilon$ for $\epsilon \in (0, \delta)$. Hint: compute the derivative with respect to $\epsilon$, recall the form of $\Delta$ when expressed in polar coordinates, and consider the radially symmetric function $\psi(r) = \int r \phi(r) \, dr$.

3. Conclude that if $g \in C_0^\infty(D)$, then the value of $(f_\epsilon, g)$ does not depend on $\epsilon$ for sufficiently small values of $\epsilon$.

4. Explain why the previous parts imply that $f \in C^\infty(D)$ and $\Delta f = 0$ (in the usual sense).

Problem 4 Suppose that $D \subseteq \mathbb{C}$ is a domain, $U \subseteq D$, and fix $f \in H(D)$.

1. Explain why there exists a unique minimizer $h \in H(D)$ to the variational problem

$$\inf \{ \| g \|_2^2 : g \in H(D), \ f|_{D \setminus U} = g|_{D \setminus U} \}.$$  

(Hint: think of this variational problem in terms of orthogonal projection.)

2. Show that $h$ is harmonic in $U$ by first showing that $(h, \Delta g) = 0$ for $g \in C_0^\infty(U)$ and then invoking the previous problem.

Problem 5 Suppose that $D \subseteq \mathbb{C}$ is a simply connected domain and fix $z \in D$. Recall that the conformal radius $C(z; D)$ of $z$ is given by $|\phi'(0)|$ where $\phi$ is any conformal transformation from $D$ to $D$ with $\phi(0) = z$. Let $G_z(y)$ be the function which is harmonic $D$ and equal to $y \mapsto -\log |z - y|$ on $\partial D$. Show that $G_z(z) = -\log C(z; D)$ using the following steps.

1. Explain why, if $f : D \to \mathbb{C}$ is a conformal map with $f(z) \neq 0$ for all $z \in D$ then $\log |f(z)|$ is harmonic in $D$.

2. Using the previous part, explain why the map $\psi : D \to \mathbb{R}$ given by

$$w \mapsto \begin{cases} \log |\phi'(w) - \phi(0)| / w & \text{for } w \neq 0 \\ \log |\phi'(0)| & \text{for } w = 0 \end{cases}$$

is harmonic in $D$.

3. Using the previous part, explain why

$$\log |\phi'(0)| = \frac{1}{2\pi} \int_{\partial D} \log |\phi(w) - \phi(0)| \, dw$$

where $dw$ denotes Lebesgue measure on $\partial D$. 

(4) Explain why \(G_z(y)\) (as defined above) is equal to \(-\psi(\phi^{-1}(y))\) and use this to explain why \(G_z(z) = -\log C(z; D)\). (Hint: use that if \(u\) is harmonic and \(\varphi\) is a conformal map then \(u \circ \varphi\) is harmonic.)

**Problem 6** Recall that if \(K \subseteq \mathbb{H}\) is a compact hull (i.e., \(\mathbb{H} \setminus K\) is simply connected and \(K\) is bounded), and \(g_K: \mathbb{H} \setminus K \to \mathbb{H}\) is the unique conformal map with \(|g_K(z) - z| \to 0\) as \(z \to \infty\), then \(\text{hcap}(K)\) is given by the coefficient of \(z^{-1}\) in the Laurent expansion

\[
g_K(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots
\]

at \(\infty\). That is, \(\text{hcap}(K) = \lim_{z \to \infty} z(g_K(z) - z)\). Show using the following steps that

\[
\text{hcap}(K) = \lim_{y \to \infty} y\mathbf{P}^{iy}[\text{Im}(B_{\tau})]
\]

where \(B\) is a standard Brownian motion in \(\mathbb{H}\), \(\mathbf{P}^{iy}\) denotes the law under which \(B_0 = y\), and \(\tau\) is the first time that \(B\) exits \(\mathbb{H} \setminus K\).

1. Let \(B_t\) be a standard Brownian motion. Explain why \(g_K(B_t) - B_t\) is a martingale.
2. Explain why \(g_K(z) - z = \mathbf{E}^z[g_K(B_{\tau}) - B_{\tau}]\) where \(\tau\) is as above.
3. Finish the proof by making the particular choice \(z = iy\) and use the representation \(\text{hcap}(A) = \lim_{z \to \infty} z(g_K(z) - z)\).

**Problem 7** Recall that a **Bessel process** of dimension \(\delta\) is given by the solution to the SDE:

\[
dx_t = \frac{\delta - 1}{2} \cdot \frac{1}{X_t} dt + dB_t, \quad X_0 > 0
\]

where \(B\) is a standard Brownian motion, at least up until the first time that \(X_t \leq 0\). Recall that \(M_t = X_t^{2-\delta}\) is a local martingale.

1. For each \(a\), let \(\tau_a = \inf\{t \geq 0 : X_t = a\}\). For \(a < X_0 < b\), compute \(\mathbf{P}[\tau_a < \tau_b]\) using that \(M\) is a local martingale.
2. Assume that \(\delta < 2\). For \(b > 1\), explain how one can condition on the event that \(\tau_b < \tau_0\) by using \(M\).
3. Using the previous part and the Girsanov theorem, describe the law of \(X|_{[0,\tau_b]}\) conditioned on \(\tau_b < \tau_0\).
4. Explain why, informally, the statement “A standard Brownian motion conditioned to be positive is a 3-dimensional Bessel process” is true.

**Problem 8** Suppose that \((g_t)\) is the chordal Loewner flow associated with an \(\text{SLE}_\kappa\) process with driving function \(W_t = \sqrt{\kappa}B_t\) and that \(\rho \in \mathbb{R}\).

1. Fix \(z \in \mathbb{H}\), let \(z_t = x_t + iy_t = g_t(z)\). Using Ito’s formula, show that \(M_t = |g'_t(z)|^{(8-2\kappa+\rho)/\kappa} y_t^{\rho^2/(8\kappa)}|W_t - z_t|^{\rho/\kappa}\) is a local martingale. (Hint: let

\[
Z_t = \frac{(8 - 2\kappa + \rho)^\rho}{8\kappa} \log g'_t(z) + \frac{\rho^2}{8\kappa} \log y_t + \frac{\rho}{\kappa} \log(W_t - z_t).
\]

Then compute \(dZ_t\), take its real part, exponentiate, and apply Ito’s formula.)

\[2\]Your answer should match the formula for \(M_t\) given in the proof [2, Theorem 6].
(2) Using the previous part and the Girsanov theorem, show that the law of \( W_{[0,t]} \) weighted by \( M_t \) is given by
\[
dW_s = \operatorname{Re} \left( \frac{\rho}{W_s - z_s} \right) ds + \sqrt{\kappa} d\tilde{B}_s
\]
where \( \tilde{B}_{[0,t]} \) is a standard Brownian motion under the law \( \tilde{P} \) weighted by \( M_t \). That is,
\[
\frac{d\tilde{P}}{dP} = \frac{M_t}{M_0}.
\]

References
