Section 1. Solutions

1A-1) a) \[ y = c_1 e^{x} + c_2 xe^{x} \]

b) \[ y' = (c_1 + c_2)e^{x} + c_2 xe^{x} \]

Add: \[ y'' - 2y' + y = 0 \] (easily checked)

c) \[ y' = \left(\frac{\sin x + a}{x^2}\right) + \cos x + \sin x \]

\[ \frac{y}{x} = \frac{\sin x + a}{x} - \frac{\cos x}{x} \]

\[ y' + \frac{y}{x} = \sin x \]

1A-3a) Separating variables gives

\[ y^2 dy = \frac{dx}{\sin x} \]

Integrate both sides from \( 2 \) to \( x \):

\[ \int_{2}^{x} y^2 \, dy = \int_{2}^{x} \frac{dt}{\sin t} \]

Now use \( y(2) = 0 \):

\[ \frac{y^3}{3} \bigg|_{2}^{x} = \int_{2}^{x} \frac{dt}{\sin t} \]

\[ y = \left[ 3 \int_{x}^{2} \frac{dt}{\sin t} \right]^{1/3} \]

b) Separate variables:

\[ \frac{dy}{y} = \frac{e^x}{x} \, dx \]

Can either use same method as \( u^2 \) (a), or else integrate both sides, using a definite integral as the antiderivative in height:

\[ \ln y + c = \int_{0}^{x} \frac{e^t}{t} \, dt \]

Evaluate \( c \) by using \( y(1) = 1 \). This gives

\[ \ln y(1) + c = \int_{0}^{1} \frac{e^t}{t} \, dt \]

\[ c = 0 \]

So,

\[ y = e^{\int_{1}^{x} \frac{e^t}{t} \, dt} \]

1A-2a) \( c_1 e^{kx} \) and \( c_2 e^{kx} \) are the same only if \( c_1 = c_2 \), \( k = k' \)

b) Let \( k = c_2 \).

Then \( y = ke^{kx} \)

c) \[ \cos 2x = \cos^2 x - \sin^2 x \]

\[ = 2\cos^2 x - 1 \]

\[ y = c_1 + c_2 (2\cos^2 x - 1) + c_3 \cos^2 x \]

\[ = (c_1 - c_2) + 2(c_2 + c_3) \cos^2 x \]

\[ = k_1 + k_2 \cos^2 x \]

d) \[ y = \ln (ax + b)(cx + d) \]

\[ = \ln (ax^2 + (ad + bd)x + bd) \]

\[ = \ln (k_1 x^2 + k_2 x + k_3) \]

1A-4a) \[ \frac{dy}{y+1} = x \, dx \]

Integrate, noting that \( \frac{y}{y+1} = 1 - \frac{1}{y+1} \)

\[ \ln y - \ln(y+1) = \frac{1}{2} x^2 \]

Put \( x = 2 \) to evaluate \( c \):

\[ 0 - \ln(1) = c + \frac{1}{2} \cdot 2 \]

\[ c = -2 \]

Solu: \[ y - \ln(y+1) = \frac{1}{2} x^2 - 2 \]

b) \[ \sec^2 u \, du = \sin t \, dt \]

\[ \tan u = -\cos t + c \]

Put \( t = 0 \):

\[ \tan 0 = -1 + c \]

\[ u(0) = 0 \]

So \( c = 1 \)

Solu: \[ u = \tan^{-1} (1 - \cos t) \]
$\frac{dy}{y^2-2y} = -\frac{dx}{x^2}$

Integrate left side by partial fractions

$\frac{1}{2} \frac{dy}{y-2} - \frac{1}{2} \frac{dy}{y} = \frac{-dx}{x^2}$

$\frac{1}{2} \ln \left( \frac{y-2}{y} \right) = c_1 + \frac{1}{x}$

Multiply by $x$; exponentiate

$\ln \left( \frac{y-2}{y} \right) = c_1 e^{2/x}$

$\Rightarrow \frac{y-2}{y} = e^{c_1 e^{2/x}}$ (algebra; replace left side by $1-\frac{2}{y}$)

$\Rightarrow y = \frac{2}{1 - c_2 e^{2/x}}$

Solve for $y$ by ordinary algebra.

$y = 1 + \frac{x+1}{1-c(x+1)}$

$\frac{dy}{\sqrt{1+x}} = \frac{dx}{x^2+4}$

$2\sqrt{1+x} = \frac{1}{2} \tan^{-1}(\frac{x}{2}) + C$

$\Rightarrow x = \frac{1}{4} \left( \frac{1}{2} \tan^{-1}(\frac{x}{2}) + C \right)^2 - 1$

These problems all take for granted that you know the standard integration formulas and methods from 18.01. Review them if you are having trouble.

You need also the laws of exponents and logarithms.
18-2

c) Divide by \( t^2 \) (so integrating factor is \( t^{-2} \))

\[
\left(1 + \frac{y}{t^2}\right) dt = \frac{x dt - t dx}{t^2}
\]

\[
\therefore \quad d\left(\frac{t - \frac{y}{t}}{t^2}\right) = d\left(\frac{\frac{t - y}{t}}{t}\right)
\]

\[
t - \frac{y}{t} = -\frac{x}{t} + c
\]

\[
x = 4 - 4t^2 + ct
\]

d) \( \frac{1}{u^2 + v^2} \) is an integrating factor:

\[
\frac{u du + v dv}{u^2 + v^2} + \frac{v du - u dv}{u^2 + v^2} = 0
\]

\[
\frac{1}{2} \ln(u^2 + v^2) + \tan^{-1}\left(\frac{u}{v}\right) = c
\]

When \( u = 0, v = 1 \), \( \frac{1}{2} \ln 1 + \tan^{-1}(0) = c \)

\[
\frac{1}{2} \ln(u^2 + v^2) + \tan^{-1}\left(\frac{u}{v}\right) = 0
\]

(substitute \( r = \sqrt{u^2 + v^2}, \theta = \tan^{-1}\frac{u}{v}\) to get polar coordinates)

equation becomes \( 2r \theta + \theta = 0 \)

\[
r = e^{-\theta}
\]

18-3

a) \( z = \frac{y}{x} \) \( \therefore \) \( y =zx \), \( y' = z'x + z \)

Substituting:

\[
z'x + z = \frac{y'x - y}{x} \quad \therefore \quad z'x = -\frac{(z+1)x}{x+y}
\]

Separate variables:

\[
\frac{z+1}{z+1} dx = \frac{\frac{dx}{x} + 1}{z+1} dx
\]

\[
\frac{(u+3)}{u^2} du = -\frac{dx}{x}
\]

Integrate:

\[
\ln u + \frac{3}{u} = -\ln x + c
\]

To improve this:

\[
\ln u x = \frac{3}{u} + c
\]

Combine and exponentiate: \( ux = ke^{\frac{3}{u}} \)

Finally:

\[
u = z + 1 = \frac{y}{x} + 1 = \frac{y+x}{x}
\]

\[
y + x = ke^{\frac{3}{y+x}}
\]

b) let \( z = \frac{y}{x} \), so \( \frac{w}{u} = \frac{z}{u} \)

Substituting:

\[
z'x + z = \frac{2z}{1-z^2}
\]

\[
z'x = \frac{(1+z^2)}{1-z^2}
\]

After some algebra

Separate variables:

\[
\frac{1-z^2}{2} dx = du \quad \therefore \quad \ln \left(\frac{\sqrt{1-z^2} + 1}{z}\right) = \ln u + c
\]

Combine and exponentiate both sides:

\[
\frac{\sqrt{1-z^2} + 1}{z} = k u
\]

Finally, put \( z = \frac{y}{x} \); result is

\[
\frac{w}{u^2 + u^2} = k
\]

as the solution

(You could also solve for \( u \) in terms of \( w \))

c) \( x = \frac{y}{x} \); so \( y = 2x, y' = 2x + 2 \)

Here \( \frac{dy}{dx} = \frac{y^2 + x^2 - y^2}{x y} \)

Substitute \( y = 2x \)

\[
\frac{z'x + z}{x} = \frac{2z + \sqrt{1-z^2}}{x}
\]

\[
\frac{z'x}{z} = \frac{\sqrt{1-z^2}}{z}
\]

Separate variables:

\[
\frac{z dz}{\sqrt{1-z^2}} = \frac{dx}{x}
\]

\[
\sqrt{1-y^2} = C_1 - bx
\]

This can be solved explicitly for \( y \):

\[
y = \frac{x}{\sqrt{1-(C_1 - bx)^2}}
\]
18-4
\[ y = u^x \]
\[ \therefore y' = x^n u' + n x^{n-1} u \]
\[ x^n u' + n x^{n-1} u = \frac{4 + x^{2n+1} u^2}{x^{n+2} u} \]
\[ \therefore u' = \frac{4 + (1-n)x^{2n+1} u^2}{x^{n+2} u} \]

If \( n = 1 \), we can separate vars:
\[ u du = \frac{4dx}{x^4} \]
\[ \therefore u^2 = \frac{-4x}{3} + c \]

Since \( n = 1 \), \( u = y/x \)
\[ \therefore \frac{y^2}{2} = \frac{-8x}{3x} + 2c x^2 \]

18-5
18-6
b) In standard form:
Integ. factor is \( e^{\int \frac{1}{x} dx} = e^{\ln(x)} = x \)
\[ x^2 y' + 2xy = x^2 \]
\[ \frac{\partial}{\partial x} \left( x^2 y \right) = \frac{\partial}{\partial x} \left( \frac{x^3}{3} + c \right) \]
\[ y = \frac{x^3}{3} + c \frac{1}{x^2} \]

18-6
b) In standard form:
Integ. factor is \( e^{\int \frac{1}{x} dx} = e^{\ln(x)} = x \)
\[ x^2 y' + 2xy = x^2 \]
\[ \frac{\partial}{\partial x} \left( x^2 y \right) = \frac{\partial}{\partial x} \left( \frac{x^3}{3} + c \right) \]
\[ y = \frac{x^3}{3} + c \frac{1}{x^2} \]

Since \( x(0) = 0 \), putting \( t = 0 \) shows \( c = 0 \).
\[ \therefore x = \frac{t^2}{2} \sec t \]

18-5

\[ (x^2-1)y' + 2xy = 1 \]
\[ \frac{(x^2-1)y'}{1} = 1 \]
\[ (x^2-1)y = x + c \]
\[ \therefore y = \frac{x+c}{x^2-1} \]

d) Writing it in standard linear form
\[ \frac{dy}{dt} + \frac{3y}{t} = 1 \]
Integrating factor: \( e^{\int \frac{3}{t} dt} = e^{3\ln t} = t^3 \)
\[ t^3 y' + 3t^2 y = t^3 \]
\[ (t^3 y)' = t^3 \]
\[ t^3 y = \frac{1}{4} t^4 + c \]
\[ y(t) = \frac{1}{4} \Rightarrow c = 0 \] \( t = 1 \)
\[ \therefore V = \frac{1}{4} \]

18-6

The integrating factor for this linear equation is \( e^{\int \frac{1}{x} dx} = e^{\ln(x)} \)
\[ (xe^{at})' = e^{at} r(t) \]
\[ x = e^{at} \left[ \int_0^t e^{as} r(s) ds \right] + c \]
\[ x = \frac{\int_0^t e^{as} r(s) ds}{e^{at}} + c \]

To find \( \lim_{t \to \infty} x(t) \), use l'Hospital's rule:
\[ \lim_{t \to \infty} \frac{\int_0^t e^{as} r(s) ds}{e^{at}} + c = 0 \] by hypothesis.

[Where did we need the hypothesis \( a > 0 \)?]

We used in conjunction with L'H rule, the result \[ \lim_{t \to \infty} \int_0^t e^{as} r(s) ds = e^{at} r(t) \]
This follows from the 2nd Fundamental Theorem of calculus.]
\[ \frac{dy}{dx} = \frac{y}{y+x} \Rightarrow \frac{dx}{dy} = \frac{u^2}{y} \]

\[ \therefore \frac{dx}{dy} - \frac{1}{y} x = y^2 \]

This is now a linear equation with \( x \).

Integ. factor: \( e^{-\int \frac{1}{y} dy} = e^{-\ln y} = y^{-1} \)

\[ \therefore \text{ multiply by } \frac{1}{y}: \]

\[ \frac{1}{y} \frac{dx}{dy} - \frac{1}{y^2} x = y \]

\[ \text{or } \frac{1}{y} \left( \frac{dy}{dx} \right) = y \]

\[ \frac{x}{y} = \frac{y^2}{2} + C \]

\[ x = \frac{y^2}{2} + Cy \]

\[ 18-8 \]

The systematic procedure - it always works, though it's a bit longer in this case. Since we want to substitute for \( y, y' \), begin by expressing them in terms of \( u \).

(Don't just differentiate \( u = y^n \) as is).

\[ y = u^{1/n} \]

\[ y' = \frac{1}{n} u^{-\frac{n-1}{n}} \cdot u' = \frac{1}{n} u^{1-n} u' \]

Substitute into the ODE:

\[ \frac{1}{n} u^{1-n} u' + pu^{1-n} = q u^{1-n} \]

Divide through by \( u^{1-n} \):

\[ \frac{1}{1-n} u' + pu = q \]

[Note: in this particular case, it's actually easier just to fumble around, but in general, this only leads to a mess.]

Hence:

\[ y' + py = q y^n \]

Divide:

\[ \frac{y'}{y^n} + \frac{p}{y^{n-1}} = q \]

Put u = \( y^n = \frac{1}{y^{n-1}} \)

\[ u' = (1-n) \cdot \frac{1}{y^{n-1}} \cdot y' \]

\[ \therefore (1-n) \cdot \frac{1}{y^{n-1}} \cdot y' \]

\[ \therefore \text{ becomes } \]

\[ \frac{u'}{1-n} + pu = q, \text{ as before.} \]

\[ \therefore \text{ becomes } \]

\[ \frac{u'}{1-n} + pu = q \]

\[ y = \pm \frac{x}{\sqrt{c-2x}} \]

\[ 18-9 \]

\[ n=2, \text{ so } u = y^{-2} = \frac{1}{y^2} \text{ (by prob. 18)} \]

Since we want to substitute for \( y, y' \), express them in terms of \( u \) and \( u' \):

\[ y = \frac{1}{u}, \quad y' = -\frac{1}{u^2} \cdot u' \]

\[ \text{ the ODE becomes } \]

\[ \frac{-u'}{u^2} + 1 = 2x \cdot \frac{1}{u^2} \]

or \[ \frac{u' - u}{u} = -2x \] in standard linear form.

Integ. factor: \( e^{\int -dx} = e^{-x} \)

Eqn becomes:

\[ (e^{-x} u)' = -2xe^{-x} \leftrightarrow \text{ integrate by parts} \]

\[ e^{-x} u = 2xe^{-x} + 2e^{-x} + C \]

\[ u = 2x + 2 + Ce^{-x} \]

\[ \boxed{u = \frac{1}{2x + 2 + Ce^{-x}}} \]
a) \[ y = y_1 + u \]
\[ y' = y_1' + u' = A + B y_1 + C y_1^2 + u' \]
Substituting into the ODE:
\[ A + B y_1 + C y_1^2 + u' = A + B(y_1 + u) + C(y_1 + u)^2 \]
After some algebra:
\[ u' = Bu + 2Cy_1 u + Cu^2 \]
\[ \therefore u' - (B + 2Cy_1) u = Cu^2 \]
This is a Bernoulli eq'n (problem 13) with \( n = 2 \).

b) By inspection, \( y_1 = x \) is a soln to the ODE. \( \therefore \) put \( y = x + u \)
\[ y' = 1 + u' \]
Substituting into the ODE gives:
\[ u' = 1 - x^2 + (x + u)^2 \]
\[ \therefore u' = -2x u = w^2 \]
a Bernoulli equation with \( n = 2 \).
Put \( w = u^{-2} = u^{-1} \)
\[ \therefore u = \frac{1}{w}, \quad u' = \frac{-w'}{w^3} \]
Substituting:
\[ -\frac{w'}{w^3} - 2x \frac{1}{w} = \frac{1}{w^2} \]
or
\[ w' + 2x w = -1 \]
Linear ODE with integrating factor
\[ e^{\int 2x \, dx} = e^{x^2} \]
\[ \therefore (e^{x^2} w)' = -e^{x^2} \]
\[ e^{x^2} w = -\int e^{x^2} \, dx + C \]
\[ W = -e^{-x^2} \int e^{x^2} \, dx + Ce^{x^2} \]
Finally:
\[ y = x + u = x + \frac{1}{W} \]
\[ \therefore y = x + \frac{e^{x^2}}{C - \int e^{x^2} \, dx} \]
(Actually, no value for \( C \) gives the original soln \( y = x \); we have to take \( C = \infty \), or simply add \( y = x \) to the above family.)
Let \( y' = z \)
\[
y'' = \frac{dz}{dx} = \frac{dz}{dy} \cdot z
\]
Substituting, \( y \cdot \frac{dz}{dy} \cdot z = z^2 \)
\[
\therefore \frac{dz}{z} = \frac{dy}{y} \quad \therefore \ln z = \ln y + C
\]
\[
\therefore \quad z = y' = Ky
\]
\[
\therefore \quad \frac{dy}{y} = K \cdot dx
\]
\[
\therefore \quad \ln y = Kx + C
\]
\[
\therefore \quad y = e^{Kx+C} \quad \therefore \text{the solution}
\]
---

(c) Let \( y' = z \)
\[
y'' = \frac{dz}{dy} \cdot z
\]
Substituting, \( \frac{dz}{dy} \cdot z = z(1+3y^2) \)
\[
\therefore \quad dz = (1+3y^2)dy
\]
\[
\therefore \quad z = y - y^3 + C \quad \text{(use the initial conditions, } C=0)
\]
\[
\therefore \quad \frac{dy}{y+y^3} = dx \quad \text{(rearrange: } z = dy \text{)}
\]
Integrating by partial fractions:
\[
\frac{1}{y+y^3} = \frac{1}{y(y^2+1)} = \frac{1}{y} - \frac{y}{y^2+1}
\]
\[
\therefore \quad \frac{dy}{y} - \frac{dy}{y^2+1} = dx
\]
\[
\ln y - \frac{1}{2} \ln(y^2+1) = x + C
\]
Exponentiating both sides,
\[
\frac{y}{\sqrt{y^2+1}} = Ke^x
\]
Using the initial conditions,
\[
\frac{1}{\sqrt{2}} = K
\]
\[
\therefore \text{solve: } \frac{y}{\sqrt{y^2+1}} = \frac{e^x}{\sqrt{2}} \quad \text{(can solve for } y \text{ in terms of } x, \text{ if desired)}
\]
\[
\therefore \quad y = \frac{e^x}{\sqrt{2-e^{2x}} \quad \text{(by squaring with sides)}}
1. Exact; also linear (divide by $\frac{dx}{y}$)

2. Linear; (integ. factor is $e^{2x}$)

3. Homogeneous: put $z=y/x$, get an ODE for $z$ where you separate variables.

4. Separate variables; also linear in $y$ and linear in $p$.

5. Exact; also linear.


7. Bernoulli equation: $n=-1$
   put $u = y^{-(n-1)} = y^2$, ...

8. Separate variables: $\frac{du}{2u} = e^{2u} du$

9. Divide by $x$ — this makes it homogeneous, so put $z=y/x$ ...

10. Linear equation (integ. factor is $\frac{1}{x^2}$)

11. Think of $y$ as dep't variable, $x$ as indep't variable; then equation is $\frac{dx}{dy} = x + y$, which is linear in $x$.

12. Separate variables; also a Bernoulli equation (exercise 13)

13. When written in the form $P(x,y) dx + Q(x,y) dy = 0$, it becomes exact.

14. Linear, with integ. factor $e^{2x}$

15. Divide by $x$ — it becomes homogeneous, so put $z = y/x$, etc.

16. Separate variable

17. Riccati equation (exercise 15a)

A particular soln is $y_1 = x^2$

make the substitution $u = y - y_1$, get Bernoulli eq'n in $u$ ($n=2$), etc.

18. Autonomous — $x$ missing.

Put $y' = v$, $y'' = v dv/du$; separate variables

19. homogeneous — put $z = x/2$

(ie: $s = ln t = ln x/2$, notice)

20. Exact when written as $Pdy + Qdx = 0$

21. Bernoulli eqn with $n=2$. (ex. 13)

22. Make change of variable $u = x + y$
   (so $u' = 1 + y'$)

Then you can separate variables

23. Becomes linear if you think $y$ as indep't variable, $x$ as dep't variable.

24. Linear (as dep't variable + indep't variable)

25. $y_1 = -x$ is a particular soln.

Riccati equation (ex. 15a) —
   put $u = y - y_1$, ...

OR BETTER:

write as $y' + (x+y)^2 + (x+y)x = 0$

and put $u = x + y$

$u' = 1 + y'$

leads to separation of variables.

26. Put $y' = v$ (so $y'' = v'$)

Get a first order linear eq'n in $v$. 

(a) Slope: \(-\frac{y}{x} = C\)

Exact solution:
\[
\frac{dy}{y} = -\frac{dx}{x}
\]

\[\ln y = -\ln x + k' \]
\[y = \frac{k'}{x}\]

(b) Slope:
\[2x + y = C\]
This is a solution.
\[y' = -2 = C\]
or \[y + 2x + 2 = 0\] is an envelope which is a solution.

(c) Slope:
\[x - y = C\]
This is a solution.
\[y' = 1 = C\]
Hence, \[x - y = 1\] is an envelope which is a solution.

(d) Slope:
\[x^2 + y^2 - 1 = C\]
To circle centre (0,0), radius \(\sqrt{1 + C}\)
**1C-1**

Isolines $y = -x - 1$ is an integral curve, so other solns cannot cross it.

**1C-2**

Isolines: $x^2 + y^2 + \frac{y}{c} = 0$, or completing the square:

$$x^2 + (y + \frac{1}{2c})^2 = (\frac{1}{2c})^2$$

(Circles, center at $(0, -\frac{1}{2c})$.)

a) decreasing, since $y' = -\frac{y}{x+y^2} < 0$

b) soln must have $y > 0$ for $x > 0$ since it cannot cross the integral curve $y = 0$.

**1C-3**

a) Using $\Delta y_n = h(x_n, y_n) = h(x_n - y_n)$, get

$$y_{n+1} = y_n + h(x_n - y_n).$$

Table entries:

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<th>$x$</th>
<th>0</th>
<th>.1</th>
<th>.2</th>
<th>.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>1</td>
<td>.9</td>
<td>.82</td>
<td>.758</td>
</tr>
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</table>

For example,

$$y_1 = y_0 + h(x_0 - y_0) = 1 + .1(-1) = .9$$

$$y_2 = y_1 + h(x_1 - y_1) = .9 + .1(1 - .9) = .82$$

$$y_3 = .82 + .1(2 - .82) = .758$$

Some isolines $x - y = c$ are drawn.

Soln curve through $(0, 1)$ is convex (= "concave up")

Thus Euler's method gives too low a result: the curve is Euler approximation.

**1C-4**

Euler method formula:

$$y_{n+1} = y_n + h f(x_n, y_n)$$

<table>
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<th>$x_n$</th>
<th>$y_n$</th>
<th>$f(x_n, y_n)$</th>
<th>$h f(x_n, y_n)$</th>
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<td>.3</td>
<td>1.463</td>
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</table>

$h = .1$

**1C-3**

b) $\Delta y_n = \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_{n+1})]

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

For this ODE, $f(x, y) = x - y$

Thus $y_{n+1}$ is the value given by the next step of Euler's method.

So, $y_0 = 1$, $y_1 = .9$ (from part a)

$$y_1 - y_0 = \frac{1}{2} [f(0, 1) + f(1, .9)]$$

$$= \frac{1}{2} [0 + -1 - .8] = -.09$$

$$y_1 = y_0 - .09$$

$$y_1 = 1 - .09 = .91$$

This does correct the Euler value ($y_1 = .9$) in the right direction, since we predicted it would be too low. (.910 is actually the correct value of the soln + 3 places.)
By the formula in 19a,
\[ y_n = y_{n-1} + h(x_{n-1} - y_{n-1}) = (1-h)y_{n-1} + hx_{n-1}. \]
But for \( x_0=0, \) we get \( x_1 = h, \)
\[ k_1 = 2h, \] and in general
\[ x_{n-1} = (n-1)h. \]

\[ y_n = (1-h)^n y_{n-1} + h^2(n-1) \]

We prove by induction that the explicit formula for \( y_n \) is:
\[ y_n = 2(1-h)^n - 1 + nh \]

a) it's true if \( n=0, \) since
\[ y_0 = 2(1-h)^0 - 1 + 0 = 1. \]

b) if true for \( y_n, \) it's true for \( y_{n+1}, \)

see, using \( \Box, \)

\[ y_{n+1} = (1-h)y_n + h^2n = 2(1-h)^n + (1-h)(1+nh) + h^2 \]

\[ y_{n+1} = (1-h)^{n+1} + (n+1)h. \]

[Note: \( \Box \) is called a difference equation. There are standard ways to solve such things; here \( \Box \) is the solution.]

Continuing, in our case \( h = \frac{1}{4}, \)

\[ y_n = 2(1-\frac{1}{4})^n - 1 + 1 = 2(1-\frac{1}{4})^n. \]

\[ \lim_{n \to \infty} y_n = 2e^{-1} \]

(since \( \lim_{k \to 0} \frac{(1-k)k}{k} = e; \))

The exact solution is
\[ y = 2e^{-x} - 1 + x, \]

so
\[ y(1) = 2e^{-1} - 1 + 1 = 2e^{-1}, \]

which checks.

It suffices to prove this is true for one step of the Runge-Kutta method and one step of Simpson's rule.

We calculate, in R-K method, the 4 slopes marked \( 1 \rightarrow 5. \)

Then we use a weighted average of them to find \( y(2h): \)

\[ y_{2h} = y_0 + 2h \cdot \frac{1}{6} \cdot (1 + 2 \cdot 4 + 2 \cdot 5 + 4) \]

Since the ODE is simply:
\[ y' = f(x), \]

from the picture

slope \( 1 = f(0), \)
slope \( 2 = f(h), \)
slope \( 3 = f(2h), \)
slope \( 4 = f(4h), \)

\[ y_{2h} = y_0 + 2h \cdot \frac{1}{6} \cdot (f(0) + 4f(h) + f(2h)) \]

Contrast this with the exact formula:
\[ y_{2h} = y_0 + \int_0^{2h} f(x) \, dx \]

Evaluating the integral approximately by one step of Simpson's rule:
\[ y_{2h} = y_0 + \frac{2h}{6} (f(0) + 4f(h) + f(2h)) \]

same as what Runge-Kutta gives.
The existence and uniqueness theorem requires the equation to be written in the form 
\[ y' = f(x, y). \]

Doing this, we get 
\[ y' = -\frac{b(x)}{c(x)} y + c(x), \]
\[ \frac{d}{dx} \]

The conclusions then are:

- \( f(x, y) \) continuous, which will be so if \( a(x), b(x), c(x) \) continuous on an interval \( [a, b, x-\delta] \) and \( a(x) \neq 0 \) on this interval.

- \( f_2(x, y) \) continuous, which will be so if \( \frac{d}{dx} \) is continuous, on and at \( x \).

But already implied by the above condition.

Note that we must have \( a(x) \neq 0 \), a condition which is often missed.

**1D-1**

- **a)**
  
  If \( (x, y) \) is a point on the curve, the geometric condition translates to:
  
  \[ \text{slope normal} = \frac{y}{x} \]
  
  \[ y' = \frac{y}{x} \]

  The solution (e.g., \( y = xe^{-x} \)) is:
  
  \[ y = c e^{\frac{x}{k}} \]

**1D-2**

- **a)**
  
  The y intercept of line \( y = mx + c \)
  
  at \( (0, c) \) is \( c = 2m \)
  
  \[ y = mx + 2m \]
  
  Differentiating \( y = mx \) \[ y' = m \]

  Eliminate \( m \):
  
  \[ y' = \frac{y}{x+2} \]

  Family

- **b)** Originated trajectories satisfy:
  
  \[ \frac{1}{y} \]
  
  \[ \frac{y}{x} \]

  \[ (x+2)^2 + y^2 = \text{constant} \]

  \[ y = \pm 1 \] are also solutions to the problem

  (above assumed implicitly that \( y \neq \pm 1 \))

- **c)**
  
  Equating slopes of normal:
  
  \[ \frac{y}{x} = -\frac{1}{y} \]

  \[ \text{Solve by exp. var., get} \]

  \[ \frac{1}{2} y^2 + x^2 = C \]

  (ellipses)

- **d)**
  
  The required property translates mathematically into:
  
  \[ \int_a^x y(t) dt = k(y(x) - y(a)) \]

  to a constant \( k \) proportionally.

  Differentiate this to get an ODE for \( y(x) \):

  \[ y(x) = y(x) \]

  \[ y'' + y = 0 \]}

  The general solution is:

  \[ y = ce^{\frac{x}{k}} \]
(b) 

\[ y = ce^x \]

\[ y' = ce^x = \frac{dy}{dx} \]

Equation of the orthogonal family:

\[ y' = -\frac{1}{y} \]

To find the curve, solve by separation of variables:

\[ y \, dy = -dx \]

\[ \frac{1}{2} y^2 = -x + C \]

Hyperbolas (all translations of one fixed hyperbola)

\[ \frac{1}{2} y^2 = -x \quad \text{along the } x \text{-axis} \]

(iii) 

\[ y' = \frac{\sqrt{x^2 + 2x^2 - 2y^2}}{2y} \]

(All tangent to } x \text{ -axis)
a) \( \frac{dx}{dt} = \text{rate of which } = \text{rate out} \),
   \[ \frac{dx}{dt} = (\text{flow in} \cdot \text{conc. in}) - (\text{flow out} \cdot \text{conc. out}) \text{ in tank} \]
   \[ x' = kc_1 - k \cdot x \]

b) \( x' + ax = 0 \) (since \( c_1 = 0 \))
   \[ x(0) = Vc_0 \]
   Solution is, by sep. p. variables,
   \[ x = Vc_0 e^{-at} \]  \( a = \frac{k}{V} \)

The general case is \( x' + ax = kc_1 \), which can be solved by separating variables, or as a linear equation.

Separating variables:
   \[ \frac{dx}{dt} = kc_1 - ax \]
   \[ \frac{dx}{kc_1 - ax} = dt \]
   \[ -\frac{1}{a} \ln(kc_1 - ax) = t + A \]  \( A \) is arbitrary constant
   or
   \[ kc_1 - ax = A e^{-at} \]

Using the initial condition to find \( A \):
   \[ kc_1 - aVc_0 = A \]  \( \text{note that } aV = k \)
   \[ A = k(c_1 - c_0) \]

so soln is (note that \( k/a = V \))
   \[ x = Vc_1 - V(c_1 - c_0)e^{-at} \]

or in terms of the concn \( c(t) \):
   \[ c = c_1 - (c_1 - c_0)e^{-at} \]

As \( t \to \infty \), \( e^{-at} \to 0 \), so \[ c \to c_1 \]

If \( c_1 = c_0 e^{-at} \), then the ODE (IVP)

\[ \begin{cases} x' + ax = kc_0 e^{-at} \\ x(0) = Vc_0 \end{cases} \]

This must be solved as a linear equation.

The integrating factor is \( e^{at} \)

or \( (xe^{at})' = kc_0 e^{(a-x)t} \)

Integrating,
   \[ xe^{at} = \frac{kC_0}{a-x} e^{at} + A \]  \( C_0 \) is constant

Using the initial condition to find \( A \):
   \[ Vc_0 = A + \frac{kC_0}{a-x} \]
   \[ x = \frac{kC_0}{a-x} e^{-at} + (Vc_0 - \frac{kC_0}{a-x}) e^{-at} \]

Dividing by \( V \) to get concn:
   \[ c = \frac{ac_0}{a-x} e^{-at} + (c_0 - \frac{ac_0}{a-x}) e^{-at} \]

[If \( a \to 0 \), then \( c = c_0 \), and this agrees with (c)]

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\[ 1D-4 \]

\[ \frac{dA}{dt} = -\lambda_1 A, \quad \lambda_1 = \text{half-life} \]

\[ \frac{dB}{dt} = \text{rate at which } B \text{ is lost by } \frac{B}{a} \text{ is lost by } \quad \frac{dB}{dt} = \lambda_2 B \]

From the first equation, \[ A = A_0 e^{-\lambda_1 t} \]

\[ \frac{dB}{dt} + \lambda_1 B = \lambda_2 A_0 e^{-\lambda_1 t} \]  \( \text{ode for } B(t) \)

Solve it as a linear equation, using \( e^{\lambda_1 t} \) as integrating factor, and \( B(0) = B_0 \) as initial condition.

Solution is:
   \[ B(t) = \frac{\lambda_2 A_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \left( B_0 - \frac{\lambda_2 A_0}{\lambda_2 - \lambda_1} \right) e^{\lambda_1 t} \]

Taking \( \lambda_1 = 1 \), \( \lambda_2 = 2 \),
   \[ B(t) = A_0 e^{-t} + (B_0 - A_0) e^{2t} \]

Differentiating \( t \) see when \( B(t) \) is maximum:
   \[ 0 = B'(t) = -A_0 e^{-t} - 2(B_0 - A_0) e^{2t} \]

Solving for \( t \):
   \[ \frac{A_0}{2(B_0 - A_0)} = e^{\lambda_1 t} \]

If \( A_0 > B_0 \), then \( t = -\ln \left( \frac{A_0}{2(B_0 - A_0)} \right) > 0 \)

If \( A_0 < B_0 \), no solution (no maximum at \( t = 0 \)).
\[ \text{By Newton's cooling law} \]
\[ \frac{dT}{dt} = K(T - 20) \quad (K \text{ a constant}) \]
Solving also (by sep. of variables) gives
\[ T(t) = \alpha e^{kt} + 20 \quad (\alpha = \text{const}) \]
\[ T(0) = 100 \]
\[ \therefore \alpha + 20 = 100 \]
\[ \therefore \alpha = 80 \]
\[ T(5) = \alpha e^{5k} + 20 = 80 \]
\[ \therefore \alpha e^{5k} = 60 \]
\[ \therefore \frac{\alpha}{e^{5k}} = 60 \]
\[ \therefore k = \frac{\ln(60/\alpha)}{5} \]
\[ \therefore T = 80 e^{-\frac{\ln(60)}{5} t} + 20 \]
When \( T = 60 \) we thus find
\[ t = \frac{\ln(60/\alpha)}{\ln(2)} = 12 \text{ mins.} \]

\[ \text{Or again: } \frac{dv}{dt} = mg - \frac{k}{m} v \]
\[ \therefore \frac{dv}{dt} + \frac{k}{m} v = g \]
\[ \text{Solving this by separation of variables (or as a linear equation), we get} \]
\[ v = \frac{mg}{K} - \frac{ma}{K} e^{-\frac{kt}{m}} \quad (a \text{ constant}) \]
\[ \text{Using the initial condition} \]
\[ v(0) = 0 \]
\[ \therefore \frac{ma}{K} + a = 0 \]
\[ \therefore v = \frac{ma}{K} (1 - e^{-\frac{kt}{m}}) \quad \text{Sol. N.} \]
\[ \text{Terminal velocity:} \]
\[ \lim_{t \to \infty} v(t) = \frac{ma}{K} \quad \text{(constant)} \]

\[ b) \quad \frac{1}{v^2} \frac{dv}{dt} = mg - \frac{k}{m} v \]
\[ \therefore \frac{dv}{v^2 - \frac{mg}{K}} = -\frac{k}{m} dt \]
But
\[ \frac{1}{v^2 - \frac{mg}{K}} = \frac{1}{v^2 - a^2} = \frac{1}{2a} \left[ \frac{1}{\sqrt{v^2 - a^2}} \right] \]
where \( a = \sqrt{\frac{mg}{K}} \)
\[ \therefore \frac{dv}{v^2 - a^2} = -\frac{2a}{m} dt \]
\[ \therefore \ln \left| \frac{v - a}{v + a} \right| = C - \frac{2at}{m} \]
But \( v(0) = 0 \)
\[ \therefore \ln 1 = C \Rightarrow C = 0 \]
\[ \therefore \frac{a - v}{a + v} = e^{-\frac{2at}{m}} \quad (\text{since } a < 0, x > 0) \]
\[ \therefore v = a \left( \frac{1 - e^{-\frac{2at}{m}}}{1 + e^{-\frac{2at}{m}}} \right) \]
\[ \lim_{t \to \infty} v(t) = \frac{a}{1 + a} = \sqrt{\frac{mg}{K}} \]
(a) Balancing forces horizontally
\[ T_0 = T_p \cos \phi = T_p \frac{dx}{ds} \]
\[ \therefore \frac{ds}{T_p} = \frac{dx}{T_0} \quad (1) \]
Balancing forces vertically
\[ W = T_p \sin \phi = T_p \frac{dy}{ds} \]
\[ \therefore \frac{ds}{T_p} = \frac{dy}{W} \quad (ii) \]

(b) Therefore the cable hangs under its own weight and has constant density \( \rho \) per unit length.

\[ W = \rho s \]

\[ \text{Now} \quad \frac{dx}{W} = \frac{dy}{W} = \frac{dy}{\rho s} \]
\[ \therefore \frac{dy}{dx} = \frac{k}{W} \quad \text{(where} \quad k = \frac{\rho}{W} \text{is a constant)} \]

\[ \text{Then} \quad \frac{ds}{dx} = k \frac{dy}{dx} = k \frac{\sqrt{(dx)^2 + (dy)^2}}{dx} \]
\[ = k \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \text{(but} \quad k > 0) \]

Also,
\[ \frac{dy}{W} = \frac{ds}{T_p} \quad \text{(from the force triangle)} \]
\[ \therefore \frac{dy}{T_p} = \frac{ds}{W s^2 + c} \]
\[ \therefore \frac{dy}{ds} = \frac{s}{\sqrt{s^2 + c}} \quad \text{where} \quad c = T_0/\rho \]
\[ \therefore y = \sqrt{s^2 + c} + c, \]
which perm (ii)

(c) Let \( \lambda \) be the constant weight per unit length.
\[ \therefore W = \lambda x \]
\[ \text{Then} \quad \frac{dy}{dx} = \frac{W}{T_0} = \frac{\lambda x}{T_0} \]
\[ \therefore y = \frac{\lambda x^2}{2T_0} + y_0 \]

Thus the cable takes the form of a parabola.

(d) Here \( W = k \) (area under \( \Theta \)).
Some rods are equally and densely spaced.
\[ \text{So} \quad \frac{dy}{dx} = \frac{W}{T_0} = \frac{k}{T_0} \int (\theta) \, dt \]
\[ \therefore \frac{dy}{dx} = k^2 y, \quad \text{by the 2nd Fund. Thm.} \]
\[ k^2 = \frac{k}{T_0} > 0 \]
The curve is one again of the form \( y = \cos(ax) + c \).

\[ 1 \text{ E - 1} \]

\[ a) \]
\[ \text{stable, unstable} \]
\[ b) \]
\[ \text{semi-stable} \]
\[ c) \]
\[ \text{unstable, stable} \]
\[ d) \]
\[ \text{stable} \]
\[ \text{(write:} \quad (x-z)^2 = -(x-z)^2) \]