3. Double Integrals

3A. Double integrals in rectangular coordinates

3A-1

a) Inner: \(6x^2y + y^2\bigg|_{y=-1}^{y=1} = 12x^2\); Outer: \(4x^3\bigg|_0^2 = 32\).

b) Inner: \(-u \cos t + \frac{1}{2}t^2 \cos u\bigg|_t=0^\pi = 2u + \frac{1}{2}\pi^2 \cos u\)

Outer: \(u^2 + \frac{1}{2}\pi^2 \sin u\bigg|_0^{\pi/2} = (\frac{1}{2}\pi)^2 + \frac{1}{2}\pi^2 = \frac{3}{4}\pi^2\).

c) Inner: \(x^2y^2\bigg|_{\sqrt{x}}^{x^2} = x^6 - x^3\); Outer: \(\frac{1}{7}x^7 - \frac{1}{4}x^4\bigg|_0^1 = \frac{1}{7} - \frac{1}{4} = -\frac{3}{28}\).

d) Inner: \(v \sqrt{u^2 + 4}\bigg|_0^u = u \sqrt{u^2 + 4}\); Outer: \(\frac{1}{3}(u^2 + 4)^{3/2}\bigg|_0^{1/3} = \frac{1}{3}(5\sqrt{5} - 8)\).

3A-2

a) (i) \(\int \int_R dy \, dx = \int_{-2}^0 \int_{-2-x}^2 dy \, dx\) (ii) \(\int \int_R dx \, dy = \int_0^2 \int_{-y}^0 dx \, dy\)

b) i) The ends of \(R\) are at 0 and 2, since \(2x - x^2 = 0\) has 0 and 2 as roots.

\(\int \int_R dy \, dx = \int_0^2 \int_{2x-x^2}^0 dy \, dx\)

ii) We solve \(y = 2x - x^2\) for \(x\) in terms of \(y\): write the equation as \(x^2 - 2x + y = 0\) and solve for \(x\) by the quadratic formula, getting \(x = 1 \pm \sqrt{1 - y}\). Note also that the maximum point of the graph is \((1, 1)\) (it lies midway between the two roots 0 and 2). We get

\(\int \int_R dx \, dy = \int_0^1 \int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} dx \, dy\).

c) (i) \(\int \int_R dy \, dx = \int_0^{\sqrt{2}} \int_0^1 dy \, dx + \int_{\sqrt{2}}^{2} \int_0^{\sqrt{1-x^2}} dy \, dx\)

(ii) \(\int \int_R dx \, dy = \int_0^{\sqrt{2}} \int_0^{\sqrt{4-y^2}} dx \, dy\)

d) Hint: First you have to find the points where the two curves intersect, by solving simultaneously \(y^2 = x\) and \(y = x - 2\) (eliminate \(x\)).

The integral \(\int \int_R dy \, dx\) requires two pieces; \(\int \int_R dx \, dy\) only one.

3A-3

a) \(\int \int_R x \, dA = \int_0^2 \int_0^{1-x^2} x \, dy \, dx\)

Inner: \(x(1 - \frac{1}{2}x)\) Outer: \(\frac{1}{2}x^2 - \frac{1}{3}x^3\bigg|_0^1 = \frac{1}{4} - \frac{8}{6} = \frac{3}{4}\).
b) \[ \int_0^1 \int_0^{1-y^2} (2x + y^2) \, dx \, dy \]
Inner: \( x^2 + y^2 x \bigg|_{y=0}^{1-y^2} = 1 - y^2 \); Outer: \( y - \frac{1}{3} y^3 \bigg|_0^1 = \frac{2}{3} \).

c) \[ \int_0^1 \int_{y-1}^{1-y} y \, dx \, dy \]
Inner: \( xy \bigg|_{y=1}^{1-y} = y[(1-y) - (y-1)] = 2y - 2y^2 \); Outer: \( y^2 - \frac{2}{3} y^3 \bigg|_0^1 = \frac{1}{3} \).

3A-4 a) \[ \int_R \int \sin^2 x \, dA = \int_{-\pi/2}^{\pi/2} \int_0^x \sin^2 x \, dy \, dx \]
Inner: \( y \sin^2 x \bigg|_0^{\pi/2} = \cos x \sin^2 x \); Outer: \( \frac{1}{3} \sin^3 x \bigg|_{-\pi/2}^{\pi/2} = \frac{1}{3}(1 - (-1)) = \frac{2}{3} \).

b) \[ \int_R \int xy \, dA = \int_0^1 \int_0^x (xy) \, dy \, dx \]
Inner: \( \frac{1}{2} xy^2 \bigg|_{x}^{x^2} = \frac{1}{2} (x^3 - x^5) \); Outer: \( \frac{1}{2} x^4 \bigg|_0^1 = \frac{1}{2} \).

c) The function \( x^2 - y^2 \) is zero on the lines \( y = x \) and \( y = -x \), and positive on the region \( R \) shown, lying between \( x = 0 \) and \( x = 1 \). Therefore
\[
\text{Volume} = \int_R \int (x^2 - y^2) \, dA = \int_0^1 \int_{-x}^x (x^2 - y^2) \, dy \, dx.
\]
Inner: \( x^2 y - \frac{1}{3} y^3 \bigg|_{-x}^x = \frac{4}{3} x^3 \); Outer: \( \frac{1}{3} x^4 \bigg|_0^1 = \frac{1}{3} \).

3A-5 a) \[ \int_0^{\pi/4} \int_0^1 e^{-y^2} \, dy \, dx = \int_0^{\pi/4} \int_0^1 e^{-y^2} \, dx \, dy = \int_0^{\pi/4} \frac{e^{-y^2}}{y} \, dy \]
\[ = \frac{1}{2} (1 - e^{-1}) \]

b) \[ \int_0^1 \int_0^1 e^{jt} \, dt \, du = \int_0^1 \int_0^1 e^{jt} \, du \, dt = \int_0^1 u e^{jt} \, du = (u - 1)e^t \bigg|_0^1 = 1 - \frac{1}{2} \sqrt{e} \]

3A-6 0; \quad \frac{1}{2} \int_S e^x \, dA, \ S = \text{upper half of } R; \quad \frac{1}{4} \int_Q x^2 \, dA, \ Q = \text{first quadrant}

0; \quad \frac{1}{4} \int_Q x^2 \, dA

3A-7 a) \( x^4 + y^4 \geq 0 \Rightarrow \frac{1}{1 + x^4 + y^4} \leq 1 \)

b) \[ \int_R \int \frac{x \, dA}{1 + x^2 + y^2} \leq \int_0^1 \int_0^1 \frac{x}{1 + x^2} \, dx \, dy = \frac{1}{2} \ln(1 + x^2) \bigg|_{y=0}^1 = \frac{\ln 2}{2} < \frac{7}{2} \]
3B. Double Integrals in polar coordinates

3B-1

a) In polar coordinates, the line \( x = -1 \) becomes \( r \cos \theta = -1 \), or \( r = -\sec \theta \). We also need the polar angle of the intersection points; since the right triangle is a 30-60-90 triangle (it has one leg 1 and hypotenuse 2), the limits are (no integrand is given):

\[
\int \int_R \, dr \, d\theta = \int_{2\pi/3} \int_{-\pi/3} \, dr \, d\theta.
\]

c) We need the polar angle of the intersection points. To find it, we solve the two equations \( r = \frac{3}{2} \) and \( r = 1 - \cos \theta \) simultaneously. Eliminating \( r \), we get \( \frac{3}{2} = 1 - \cos \theta \), from which \( \theta = \frac{2\pi}{3} \) and \( \frac{4\pi}{3} \). Thus the limits are (no integrand is given):

\[
\int \int_R \, dr \, d\theta = \int_{\pi/3}^{\pi/3} \int_{-\pi/3}^{\pi/3} \, dr \, d\theta.
\]

d) The circle has polar equation \( r = 2a \cos \theta \). The line \( y = a \) has polar equation \( r \sin \theta = a \), or \( r = a \csc \theta \). Thus the limits are (no integrand):

\[
\int \int_R \, dr \, d\theta = \int_{\pi/4}^{\pi/4} \int_{\pi/2}^{\pi/2} \, dr \, d\theta.
\]

3B-2

a) \[ \int_0^{\pi/2} \int_0^{\sin \theta} \frac{r \, dr \, d\theta}{r} = \int_0^{\pi/2} \sin 2\theta \, d\theta = -\frac{1}{2} \cos 2\theta \bigg|_0^{\pi/2} = -\frac{1}{2}(-1) = 1. \]

b) \[ \int_0^\alpha \int_0^r \frac{1}{1+r^2} \, dr \, d\theta = \frac{\pi}{2} \cdot \frac{1}{2} \ln(1+r^2) \bigg|_0^\alpha = \frac{\pi}{4} \ln(1+a^2). \]

c) \[ \int_0^{\pi/4} \int_0^{\sec \theta} \tan^2 \theta \cdot r \, dr \, d\theta = \frac{1}{2} \int_0^{\pi/4} \tan^2 \theta \, d\theta = \frac{1}{6} \tan^3 \theta \bigg|_0^{\pi/4} = \frac{1}{6}. \]

d) \[ \int_0^{\pi/2} \int_0^{\sin \theta} \frac{r \, dr \, d\theta}{\sqrt{1-r^2}} \]

Inner: \(-\sqrt{1-r^2}\sin \theta \bigg|_0 = 1 - \cos \theta\) Outer: \(\theta - \sin \theta \bigg|_0^{\pi/2} = \pi/2 - 1.\)

3B-3

a) the hemisphere is the graph of \( z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - r^2} \), so we get
\[ \int_R \sqrt{a^2 - r^2} \, dA = \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2} \, r \, dr \, d\theta = 2\pi \left( \frac{1}{3} \right) \left( \frac{a}{2} \right)^3 = \frac{2}{3} \pi a^3. \]

b) \[ \int_0^{\pi/2} \int_0^a (r \cos \theta)(r \sin \theta) \, dr \, d\theta = \int_0^a r^3 \, dr \int_0^{\pi/2} \sin \theta \cos \theta \, d\theta = \frac{a^4}{4} \cdot \frac{1}{2} = \frac{a^4}{8}. \]

c) In order to be able to use the integral formulas at the beginning of 3B, we use symmetry about the y-axis to compute the volume of just the right side, and double the answer.

\[ \int_R \sqrt{x^2 + y^2} \, dA = 2 \int_0^{\pi/2} \int_0^a r^2 \, r \, dr \, d\theta = 2 \int_0^{\pi/2} \frac{1}{3} (2 \sin \theta)^3 \, d\theta \]

\[ = 2 \cdot \frac{8}{3} \cdot \frac{2}{3} = \frac{32}{9}, \text{ by the integral formula at the beginning of 3B.} \]

d) \[ \int_0^{\pi/2} \int_0^a r^2 \, r \, dr \, d\theta = 2 \int_0^{\pi/2} \frac{1}{4} \cos^2 \theta \, d\theta = 2 \cdot \frac{1}{4} \cdot \frac{\pi}{4} = \frac{\pi}{8}. \]

3C. Applications of Double Integration

3C-1 Placing the figure so its legs are on the positive x- and y-axes,

a) M.I. = \[ \int_0^a \int_0^{a-x} x^2 \, dy \, dx \] Inner: \( x^2y \big|_0^{a-x} = x^2(a-x); \) Outer: \[ \left[ \frac{1}{3} x^3 a - \frac{1}{4} a^4 \right]_0^a = \frac{1}{12} a^4. \]

b) \[ \int_R (x^2 + y^2) \, dA = \int_R x^2 \, dA + \int_R y^2 \, dA = \frac{1}{12} a^4 + \frac{1}{12} a^4 = \frac{1}{6} a^4. \]

c) Divide the triangle symmetrically into two smaller triangles, their legs are \( \frac{a}{\sqrt{2}}; \)

Using the result of part (a), M.I. of \( R \) about hypotenuse = \[ 2 \cdot \frac{1}{12} \left( \frac{a}{\sqrt{2}} \right)^4 = \frac{a^4}{24} \]

3C-2 In both cases, \( \bar{x} \) is clear by symmetry; we only need \( \bar{y}. \)

a) Mass is \( \int \int_R \, dA = \int_0^\pi \sin x \, dx = 2 \)

y-moment is \( \int \int_R y \, dA = \int_0^\pi \int_0^{\sin x} y \, dy \, dx = \frac{1}{2} \int_0^\pi \sin^2 x \, dx = \frac{\pi}{4}; \) therefore \( \bar{y} = \frac{\pi}{8}. \)

b) Mass is \( \int \int_R y \, dA = \frac{\pi}{4}, \) by part (a). Using the formulas at the beginning of 3B,

y-moment is \( \int \int_R y^2 \, dA = \int_0^\pi \int_0^{\sin x} y^2 \, dy \, dx = 2 \int_0^{\pi/2} \sin^3 x \, dx = 2 \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9}. \)

Therefore \( \bar{y} = \frac{4}{9} \cdot \frac{4}{\pi} = \frac{16}{9\pi}. \)
3C-3 Place the segment either horizontally or vertically, so the diameter is respectively on the $x$ or $y$ axis. Find the moment of half the segment and double the answer.

(a) (Horizontally, using rectangular coordinates) Note that $a^2 - c^2 = b^2$.

$$
\int_0^b \int_c^{\sqrt{a^2-x^2}} y \, dy \, dx = \int_0^b \frac{1}{2} (a^2-x^2) \, dx = \frac{1}{2} \left[ b^2 x - \frac{x^3}{3} \right]_0 = \frac{1}{3} b^3; \quad \text{ans: } \frac{2}{3} b^3.
$$

(b) (Vertically, using polar coordinates). Note that $x = c$ becomes $r = c \sec \theta$.

Moment = $\int_0^\alpha \int_0^a (r \cos \theta) \, r \, dr \, d\theta$

Inner: $\frac{2}{3} a^3 \sin \theta \bigg|_0^\alpha = \frac{2}{3} (a^2 b - c^2 b) = \frac{2}{3} b^3$; \quad \text{ans: } \frac{2}{3} b^3.

3C-4 Place the sector so its vertex is at the origin and its axis of symmetry lies along the positive $x$-axis. By symmetry, the center of mass lies on the $x$-axis, so we only need find $\bar{x}$.

Since $\delta = 1$, the area and mass of the disc are the same: $\pi a^2 \cdot \frac{2a}{2\pi} = a^2 \alpha$.

$x$-moment: $2 \int_0^\alpha \int_0^a r \cos \theta \cdot r \, dr \, d\theta$

Inner: $\frac{2}{3} a^3 \cos \theta \bigg|_0^\alpha = \frac{2}{3} (a^2 b - c^2 b)$; \quad Outer: $\frac{2}{3} a^3 \sin \theta \bigg|_0^\alpha = \frac{2}{3} a \cdot \sin \alpha \cdot \frac{a^2 \alpha}{a^2}$

$\bar{x} = \frac{2}{3} \cdot \frac{a^3 \sin \alpha}{a^2 \alpha} = \frac{2}{3}, \quad \text{ans: } \frac{2}{3} a^2 \alpha$.

3C-5 By symmetry, we use just the upper half of the loop and double the answer. The upper half lies between $\theta = 0$ and $\theta = \pi/4$.

$$
2 \int_0^{\pi/4} \int_0^{a \sqrt{\cos 2\theta}} r^2 \, r \, dr \, d\theta = 2 \int_0^{\pi/4} r^2 \cos^2 2\theta \, d\theta
$$

Putting $u = 2\theta$, the above = $\frac{a^4}{2} \int_0^{\pi/2} \cos^2 u \, du = \frac{a^4}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi a^4}{16}$. \quad $r = a \sqrt{\cos 2\theta}$

3D. Changing Variables

3D-1 Let $u = x - 3y$, \quad $v = 2x + y$;

$$
\frac{\partial (u,v)}{\partial (x,y)} = \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = 7; \quad \frac{\partial (x,y)}{\partial (u,v)} = \frac{1}{7}.
$$

$$
\int \int_R \frac{x-3y}{2x+y} \, dx \, dy = \frac{1}{7} \int_0^7 \int_1^4 \frac{u}{v} \, dv \, du
$$

Inner: $u \ln v \bigg|_1^4 = u \ln 4$; \quad Outer: $\frac{1}{2} u^2 \ln 4 \bigg|_0^7 = \frac{49 \ln 4}{2}$; \quad Ans: $\frac{49 \ln 4}{2} = 7 \ln 2$.
3D-2 Let \( u = x + y, \ v = x - y. \) Then \( \frac{\partial(u,v)}{\partial(x,y)} = 2; \ \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}. \)

To get the \( uv \)-equation of the bottom of the triangular region:
\[
y = 0 \quad \Rightarrow \quad u = x, \ v = x \quad \Rightarrow \quad u = v.
\]

\[
\int \int_R \cos \left( \frac{x-y}{x+y} \right) \, dx \, dy = \frac{1}{2} \int_0^\infty \int_0^v \cos \frac{u}{v} \, dv \, du
\]

Inner: \( u \sin \frac{v}{u} \bigg|_{u=0}^v = u \sin 1 \) Outer: \( \frac{1}{2} u^2 \sin 1 \bigg|_0^v = 2 \sin 1 \) Ans: \( \sin 1 \)

3D-3 Let \( u = x, \ v = 2y; \) then \( \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2} \left[ \begin{array}{l} 1 \\ 0 \\ \frac{1}{2} \end{array} \right] = \frac{1}{2} \)

Letting \( R \) be the elliptical region whose boundary is \( x^2 + 4y^2 = 16 \) in \( xy \)-coordinates, and \( u^2 + v^2 = 16 \) in \( uv \)-coordinates (a circular disc), we have
\[
\int \int_R (16 - x^2 - 4y^2) \, dy \, dx = \frac{1}{2} \int \int_R (16 - u^2 - v^2) \, dv \, du
\]
\[
= \frac{1}{2} \int_0^{2\pi} \int_0^4 (16 - r^2) \, r \, dr \, d\theta = \pi \left( 16 \int_0^4 r^2 - \frac{r^4}{4} \right) = 64\pi.
\]

3D-4 Let \( u = x + y, \ v = 2x - 3y; \) then \( \frac{\partial(u,v)}{\partial(x,y)} = \left| \begin{array}{l} 1 \\ 2 \\ -3 \end{array} \right| = -5; \ \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{5} \)

We next express the boundary of the region \( R \) in \( uv \)-coordinates.
For the \( x \)-axis, we have \( y = 0, \) so \( u = x, \ \ v = 2x, \) giving \( v = 2u. \)
For the \( y \)-axis, we have \( x = 0, \) so \( u = y, \ \ v = -3y, \) giving \( v = -3u. \)

It is best to integrate first over the lines shown, \( v = c; \) this means \( v \) is held constant, i.e., we are integrating first with respect to \( u. \) This gives
\[
\int \int_R (2x - 3y)^2(x + y)^2 \, dx \, dy = \int_0^{v/2} \int_{-\sqrt{v/2}}^{\sqrt{v/2}} v^2 u^2 \, du \, dv.
\]

Inner: \( \frac{v^2 u^2}{15} \bigg|_{-\sqrt{v/2}}^{\sqrt{v/2}} = \frac{v^2}{27} \left( \frac{1}{8} - \frac{1}{27} \right) \) Outer: \( \frac{v^6}{6 \cdot 15} \left( \frac{1}{8} + \frac{1}{27} \right) \bigg|_0^{-\sqrt{v/2}} = \frac{4^6}{6 \cdot 15} \left( \frac{1}{8} + \frac{1}{27} \right). \)

3D-5 Let \( u = xy, \ v = y/x; \) in the other direction this gives \( y^2 = uv, \ x^2 = u/v. \)

We have \( \frac{\partial(u,v)}{\partial(x,y)} = \left| \begin{array}{l} y \\ -y/x^2 \\ 1/x \end{array} \right| = 2y \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v}; \) this gives
\[
\int \int_R (x^2 + y^2) \, dx \, dy = \int_0^{v/2} \int_0^{u/v} \left( \frac{u}{v} + w \right) \, dv \, du.
\]

Inner: \( \frac{-u}{2v} + \frac{u}{2} \bigg|_0^{v/2} = \left( \frac{1}{4} + 1 - 2 - \frac{1}{2} \right) = \frac{3u}{4}; \) Outer: \( \frac{3u^3}{8} \bigg|_0^{v/2} = \frac{27}{8}. \)

3D-8 a) \( y = x^2; \) therefore \( u = x^3, \ v = x, \) which gives \( u = v^3. \)
b) We get \( \frac{u}{v} + uv = 1 \), or \( u = \frac{v}{v^2 + 1} \); (cf. 3D-5)
18.02 Notes and Exercises by A. Mattuck and Bjorn Poonen with the assistance of T. Shifrin and S. LeDuc

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