2. Partial Differentiation

2A. Functions and Partial Derivatives

2A-1 In the pictures below, not all of the level curves are labeled. In (c) and (d), the picture is the same, but the labelings are different. In more detail:
   b) the origin is the level curve 0; the other two unlabeled level curves are .5 and 1.5;
   c) on the left, two level curves are labeled; the unlabeled ones are 2 and 3; the origin is the level curve 0;
   d) on the right, two level curves are labeled; the unlabeled ones are −1 and −2; the origin is the level curve 1;
   The crude sketches of the graph in the first octant are at the right.

2A-2 a) \( f_x = 3x^2y - 3y^2, \quad f_y = x^3 - 6xy + 4y \)
   b) \( z_x = \frac{1}{y}, \quad z_y = -\frac{x}{y^2} \)
   c) \( f_x = 3\cos(3x + 2y), \quad f_y = 2\cos(3x + 2y) \)
   d) \( f_x = 2xye^{xy}, \quad f_y = x^2e^{xy} \)
   e) \( z_x = \ln(2x + y) + \frac{2x}{2x + y}, \quad z_y = \frac{x}{2x + y} \)
   f) \( f_x = 2xz, \quad f_y = -2z^3, \quad f_z = 2x - 6yz^2 \)

2A-3 a) both sides are \( mnx^{m-1}y^{n-1} \)
   b) \( f_x = \frac{y}{(x+y)^2}, \quad f_{xy} = (f_x)_y = \frac{x-y}{(x+y)^3}; \quad f_y = \frac{-x}{(x+y)^2}, \quad f_{yx} = \frac{(y-x)}{(x+y)^3}. \)
   c) \( f_x = -2x\sin(x^2 + y), \quad f_{xy} = (f_x)_y = -2x\cos(x^2 + y); \)
      \( f_y = -\sin(x^2 + y), \quad f_{yx} = -\cos(x^2 + y) \cdot 2x. \)
   d) both sides are \( f'(x)g'(y) \).

2A-4 \( (f_x)_y = ax + 6y, \quad (f_y)_x = 2x + 6y; \) therefore \( f_{xy} = f_{yx} \iff a = 2. \) By inspection, one sees that if \( a = 2, \quad f(x, y) = x^2y + 3xy^2 \) is a function with the given \( f_x \) and \( f_y. \)

2A-5 a) \( w_x = ae^{ax}\sin ay, \quad w_{xx} = a^2e^{ax}\sin ay; \)
   \( w_y = e^{ax}a\cos ay, \quad w_{yy} = e^{ax}a^2(-\sin ay); \) therefore \( w_{yy} = -w_{xx}. \)
   b) We have \( w_x = \frac{2x}{x^2 + y^2}, \quad w_{xx} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}. \) If we interchange \( x \) and \( y, \) the function \( w = \ln(x^2 + y^2) \) remains the same, while \( w_{xx} \) gets turned into \( w_{yy}; \) since the interchange just changes the sign of the right hand side, it follows that \( w_{yy} = -w_{xx}. \)

2B. Tangent Plane; Linear Approximation

2B-1 a) \( z_x = y^2, \quad z_y = 2xy; \) therefore at (1,1,1), we get \( z_x = 1, \quad z_y = 2, \) so that the tangent plane is \( z = 1 + (x - 1) + 2(y - 1), \) or \( z = x + 2y - 2. \)
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b) \( w_x = -y^2/x^2, \quad w_y = 2y/x; \) therefore at (1,2,4), we get \( w_x = -4, \quad w_y = 4, \) so that the tangent plane is \( w = 4 - 4(x - 1) + 4(y - 2), \) or \( w = -4x + 4y. \)

2B-2 a) \( z_x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{z}; \) by symmetry (interchanging \( x \) and \( y \)), \( z_y = \frac{y}{z}; \) then the tangent plane is \( z = z_0 + \frac{x_0}{z_0} (x - x_0) + \frac{y_0}{z_0} (y - y_0), \) or \( z = \frac{x_0}{z_0} x + \frac{y_0}{z_0} y, \) since \( x_0^2 + y_0^2 = z_0^2. \)

b) The line is \( x = x_0 t, \quad y = y_0 t, \quad z = z_0 t; \) substituting into the equations of the cone and the tangent plane, both are satisfied for all values of \( t; \) this shows the line lies on both the cone and tangent plane (this can also be seen geometrically).

2B-3 Letting \( x, y, z \) be respectively the lengths of the two legs and the hypotenuse, we have \( z = \sqrt{x^2 + y^2}; \) thus the calculation of partial derivatives is the same as in 2B-2, and we get \( \Delta z \approx \frac{3}{5} \Delta x + \frac{4}{5} \Delta y. \) Taking \( \Delta x = \Delta y = .01, \) we get \( \Delta z \approx \frac{7}{5}(.01) = .014. \)

2B-4 From the formula, we get \( R = \frac{R_1 R_2}{R_1 + R_2}. \) From this we calculate

\[
\frac{\partial R}{\partial R_1} = \left( \frac{R_2}{R_1 + R_2} \right)^2, \quad \text{and by symmetry,} \quad \frac{\partial R}{\partial R_2} = \left( \frac{R_1}{R_1 + R_2} \right)^2.
\]

Substituting \( R_1 = 1, \quad R_2 = 2 \) the approximation formula then gives \( \Delta R = -\frac{4}{9} \Delta R_1 + \frac{1}{9} \Delta R_2. \)

By hypothesis, \( |\Delta R_i| \leq .1, \) for \( i = 1, 2, \) so that \( |\Delta R| \leq \frac{4}{9}(.1) + \frac{1}{9}(.1) = \frac{5}{9}(.1) \approx .06; \) thus

\( R = \frac{2}{3} = .67 \pm .06. \)

2B-5 a) We have \( f(x, y) = (x+y+2)^2, \quad f_x = 2(x+y+2), \quad f_y = 2(x+y+2). \) Therefore

at \( (0, 0), \quad f_x(0, 0) = f_y(0, 0) = 4, \quad f(0, 0) = 4; \) linearization is \( 4 + 4x + 4y; \)

at \( (1, 2), \quad f_x(1, 2) = f_y(1, 2) = 10, \quad f(1, 2) = 25; \) linearization is \( 10(x-1) + 10(y-2) + 25, \) or \( 10x + 10y - 5. \)

b) \( f = e^x \cos y; \quad f_x = e^x \cos y; \quad f_y = -e^x \sin y \).

Linearization at \( (0, 0): \quad 1 + x; \) linearization at \( (0, \pi/2): \quad -(y - \pi/2) \)

2B-6 We have \( V = \pi r^2 h, \quad \frac{\partial V}{\partial r} = 2\pi rh, \quad \frac{\partial V}{\partial h} = \pi r^2; \quad \Delta V \approx \left( \frac{\partial V}{\partial r} \right)_0 \Delta r + \left( \frac{\partial V}{\partial h} \right)_0 \Delta h. \)

Evaluating the partials at \( r = 2, \quad h = 3, \) we get

\( \Delta V \approx 12\pi \Delta r + 4\pi \Delta h. \)

Assuming the same accuracy \( |\Delta r| \leq \epsilon, \quad |\Delta h| \leq \epsilon \) for both measurements, we get

\( |\Delta V| \leq 12\pi \epsilon + 4\pi \epsilon = 16\pi \epsilon, \) which is \( < .1 \) if \( \epsilon < \frac{1}{160\pi} < .002. \)

2B-7 We have \( r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \frac{y}{x}; \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}. \)

Therefore at \( (3, 4), \quad r = 5, \) and \( \Delta r \approx \frac{3}{5} \Delta x + \frac{4}{5} \Delta y. \) If \( |\Delta x| \) and \( |\Delta y| \) are both \( \leq .01, \) then

\( |\Delta r| \leq \frac{3}{5} |\Delta x| + \frac{4}{5} |\Delta y| = \frac{7}{5}(.01) = .014 \) (or .02).

Similarly, \( \frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2}; \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}, \) so at the point \( (3, 4), \)
\[|\Delta \theta| \leq \left| \frac{2}{350} \Delta x \right| + \left| \frac{2}{350} \Delta y \right| \leq \frac{2}{350}(0.01) = 0.0028 \text{ (or } 0.003).\]

Since at (3, 4) we have \(|r_y| > |r_x|\), \(r\) is more sensitive there to changes in \(y\); by analogous reasoning, \(\theta\) is more sensitive there to \(x\).

2B-9 a) \(w = x^2(y + 1)\); \(w_x = 2x(y + 1) = 2\) at (1, 0), and \(w_y = x^2 = 1\) at (1, 0); therefore \(w\) is more sensitive to changes in \(x\) around this point.

b) To first order approximation, \(\Delta w \approx 2\Delta x + \Delta y\), using the above values of the partial derivatives.

If we want \(\Delta w = 0\), then by the above, \(2\Delta x + \Delta y = 0\), or \(\Delta y/\Delta x = -2\).

2C. Differentials; Approximations

2C-1 a) \(dw = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}\)  
b) \(dw = 3x^2y^2z\,dx + 2x^3yz\,dy + x^3y^2\,dz\)

c) \(dz = \frac{2y\,dx - 2x\,dy}{(x + y)^2}\)  
d) \(dw = \frac{t\,du - u\,dt}{\sqrt{t^2 - u^2}}\)

2C-2 The volume is \(V = xyz\); so \(dV = yz\,dx + xz\,dy + xy\,dz\). For \(x = 5\), \(y = 10\), \(z = 20\),
\[\Delta V \approx dV = 200\,dx + 100\,dy + 50\,dz,\]
from which we see that \(|\Delta V| \leq 350(0.1); \text{ therefore } V = 1000 \pm 35.\)

2C-3 a) \(A = \frac{1}{2}ab\sin \theta\). Therefore, \(dA = \frac{1}{2}(b\sin \theta\,da + a\sin \theta\,db + ab\cos \theta\,d\theta)\).

b) \(dA = \frac{1}{2}(2(1)\,da + 1(1)\,db + 1(2) \frac{1}{2}\sqrt{3}\,d\theta) = \frac{1}{2}(da + \frac{1}{2}db + \sqrt{3}\,d\theta)\); therefore most sensitive to \(\theta\), least sensitive to \(b\), since \(d\theta\) and \(db\) have respectively the largest and smallest coefficients.

c) \(dA = \frac{1}{2}(0.02 + 0.01 + 1.73)(0.02) \approx \frac{1}{2}(0.065) \approx .03\)

2C-4 a) \(P = \frac{kT}{V}\); therefore \(dP = \frac{k}{V}dT - \frac{kT}{V^2}dV\)

b) \(V\,dP + P\,dV = k\,dT\); therefore \(dP = \frac{k}{V}dT - P\,dV\).

c) Substituting \(P = kT/V\) into (b) turns it into (a).

2C-5 a) \(-\frac{dw}{w^2} = -\frac{dt}{t^2} - \frac{du}{u^2} - \frac{dv}{v^2}; \text{ therefore } dw = u^2\left(\frac{dt}{t^2} + \frac{du}{u^2} + \frac{dv}{v^2}\right)\).

b) \(2u\,du + 4v\,dv + 6w\,dw = 0; \text{ therefore } dw = -\frac{u\,du + 2v\,dv}{3w}\).

2D. Gradient; Directional Derivative

2D-1 a) \(\nabla f = 3x^2i + 6y^2j\); \(\left(\nabla f\right)_P = 3i + 6j\); \(\left.\frac{df}{ds}\right|_u = \left(3i + 6j\right) \cdot \frac{i - j}{\sqrt{2}} = -\frac{3\sqrt{2}}{2}\)

b) \(\nabla w = \frac{y}{z}i + \frac{x}{z}j - \frac{xy}{z^2}k\); \(\left(\nabla w\right)_P = -i + 2j + 2k\); \(\left.\frac{dw}{ds}\right|_u = \left(\nabla w\right)_P \cdot \frac{i + 2j - 2k}{3} = -\frac{1}{3}\)

c) \(\nabla z = (\sin y - y \sin x)i + (x \cos y + \cos x)j\); \(\left(\nabla z\right)_P = i + j\); \(\frac{dz}{ds}|_u = (i + j) \cdot \frac{-3i + 4j}{5} = \frac{1}{5}\).
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2D-2

d) \( \nabla w = \frac{2i + 3j}{2t + 3u} \); \( (\nabla w)_P = 2i + 3j \);
\[ \frac{dw}{ds} \bigg|_u = (2i + 3j) \cdot \frac{4i - 3j}{5} = -\frac{1}{5} \]

e) \( \nabla f = 2(u + 2v + 3w)(i + 2j + 3k) \); \( (\nabla f)_P = 4(i + 2j + 3k) \)
\[ \frac{df}{ds} \bigg|_u = 4(i + 2j + 3k) \cdot \frac{-2i + 2j - k}{3} = \frac{4}{3} \]

2D-3

a) \( \nabla w = \frac{4i - 3j}{4x - 3y} \); \( (\nabla w)_P = 4i - 3j \)
\[ \frac{dw}{ds} \bigg|_u = (4i - 3j) \cdot \mathbf{u} \text{ has maximum 5, in the direction } \mathbf{u} = \frac{4i - 3j}{5}, \]
and minimum \(-5\) in the opposite direction.
\[ \frac{dw}{ds} \bigg|_u = 0 \text{ in the directions } \pm \frac{3i + 4j}{5}. \]

b) \( \nabla w = \langle y + z, x + z, x + y \rangle \); \( (\nabla w)_P = (1, 3, 0) \)
\[ \max \frac{dw}{ds} \bigg|_u = \sqrt{10}, \text{ direction } \frac{i + 3j}{\sqrt{10}}; \quad \min \frac{dw}{ds} \bigg|_u = -\sqrt{10}, \text{ direction } -\frac{i + 3j}{\sqrt{10}}; \]
\[ \frac{dw}{ds} \bigg|_u = 0 \text{ in the directions } \mathbf{u} = \pm \frac{-3i + j + ck}{\sqrt{10} + c^2} \text{ (for all } c) \]

c) \( \nabla w = 2\sin(t - u) \cos(t - u)(i - j) \); \( (\nabla w)_P = i - j \)
\[ \max \frac{dw}{ds} \bigg|_u = \sqrt{2}, \text{ direction } \frac{i - j}{\sqrt{2}}; \quad \min \frac{dw}{ds} \bigg|_u = -\sqrt{2}, \text{ direction } -\frac{i + j}{\sqrt{2}}; \]
\[ \frac{dw}{ds} \bigg|_u = 0 \text{ in the directions } \pm \frac{i + j}{\sqrt{2}}. \]

2D-4

a) \( \nabla T = \frac{2x i + 2y j}{x^2 + y^2} \); \( (\nabla T)_P = \frac{2i + 4j}{5} \)
\( T \text{ is increasing at } P \text{ most rapidly in the direction of } (\nabla T)_P, \text{ which is } \frac{i + 2j}{\sqrt{5}}. \)

b) \( |\nabla T| = \frac{2}{\sqrt{5}} \) is rate of increase in direction \( \frac{i + 2j}{\sqrt{5}}. \) Call the distance to go \( \Delta s, \) then
\[ \frac{2}{\sqrt{5}} \Delta s = .20 \quad \Rightarrow \quad \Delta s = \frac{2\sqrt{5}}{2} = \frac{\sqrt{5}}{10} \approx .22. \]

c) \( \frac{dT}{ds} \bigg|_u = (\nabla T)_P \cdot \mathbf{u} = \frac{2i + 4j}{5} \cdot \frac{i + j}{\sqrt{2}} = \frac{6}{5\sqrt{2}}; \)
\[ \frac{6}{5\sqrt{2}} \Delta s = .12 \quad \Rightarrow \quad \Delta s = \frac{5\sqrt{2}}{6}(.12) \approx (.10)(\sqrt{2}) \approx .14 \]

d) In the directions orthogonal to the gradient: \( \pm \frac{2i - j}{\sqrt{5}} \)
2D-5  a) isotherms = the level surfaces \( x^2 + 2y^2 + 2z^2 = c \), which are ellipsoids.

b) \( \nabla T = (2x, 4y, 4z); \quad (\nabla T)_p = (2, 4, 4); \quad |(\nabla T)_p| = 6; \)

for most rapid decrease, use direction of \(- (\nabla T)_p : -\frac{1}{3} (1, 2, 2)\)

c) let \( \Delta s \) be distance to go; then \(-6(\Delta s) = -1.2; \quad \Delta s \approx .2 \)

d) \( \frac{dT}{ds} \bigg|_u = (\nabla T)_p \cdot u = (2, 4, 4) \cdot \left\langle 1, -\frac{2}{3}, \frac{1}{3} \right\rangle = \frac{2}{3}; \quad \frac{2}{3} \Delta s \approx .10 \Rightarrow \Delta s \approx .15. \)

2D-6 \( \nabla uv = (uv)_x, (uv)_y = (uv_x + vu_x, uv_y + vu_y) = (uv_x, uv_y + vu_x + vu_y) = u\nabla v + v\nabla u \)

\[\nabla (uv) = u\nabla v + v\nabla u \Rightarrow \nabla (uv) \cdot u = u\nabla v \cdot u + v\nabla u \cdot u \Rightarrow \frac{d(uv)}{ds} \bigg|_u = u \frac{dv}{ds} \bigg|_u + v \frac{du}{ds} \bigg|_u.\]

2D-7 At \( P \), let \( \nabla w = a \hat{i} + b \hat{j} \). Then

\[\begin{align*}
  a\hat{i} + b\hat{j} \cdot \frac{i + j}{\sqrt{2}} &= 2 \Rightarrow a + b = 2\sqrt{2} \\
  a\hat{i} + b\hat{j} \cdot \frac{i - j}{\sqrt{2}} &= 1 \Rightarrow a - b = \sqrt{2}
\end{align*}\]

Adding and subtracting the equations on the right, we get \( a = \frac{3}{2} \sqrt{2}, \quad b = \frac{1}{2} \sqrt{2}. \)

2D-8 We have \( P(0, 0, 0) = 32 \); we wish to decrease it to 31.1 by traveling the shortest distance from the origin \( 0 \); for this we should travel in the direction of \(- (\nabla P)_0 \).

\( \nabla P = \left\langle (y + 2)e^z, (x + 1)e^z, (x + 1)(y + 2)e^z \right\rangle; \quad (\nabla P)_0 = (2, 1, 2). \quad |(\nabla P)_0| = 3. \)

Since \(-3 \cdot (\Delta s) = -9 \Rightarrow \Delta s = .3, \) we should travel a distance .3 in the direction of \(- (\nabla P)_0 \). Since \| - (2, 1, 2) \| = 3, the distance .3 will be \( \frac{1}{10} \) of the distance from \( (0, 0, 0) \) to \( (-2, -1, -2) \), which will bring us to \( (-2, -1, -2) \).

2D-9 In these, we use \( \frac{dw}{ds} \bigg|_u \approx \frac{\Delta w}{\Delta s} \): we travel in the direction \( u \) from a given point \( P \) to the nearest level curve \( C \); then \( \Delta s \) is the distance traveled (estimate it by using the unit distance), and \( \Delta w \) is the corresponding change in \( w \) (estimate it by using the labels on the level curves).

a) The direction of \( \nabla f \) is perpendicular to the level curve at \( A \), in the increasing sense (the “uphill” direction). The magnitude of \( \nabla f \) is the directional derivative in that direction: from the picture, \( \frac{\Delta w}{\Delta s} \approx \frac{1}{.5} = 2. \)

b), c) \( \frac{\partial w}{\partial x} \bigg|_i = \frac{dw}{ds} \bigg|_i, \quad \frac{\partial w}{\partial y} \bigg|_j = \frac{dw}{ds} \bigg|_j, \) so \( B \) will be where \( i \) is tangent to the level curve and \( C \) where \( j \) is tangent to the level curve.

d) At \( P \), \( \frac{\partial w}{\partial x} \bigg|_i \approx \frac{\Delta w}{\Delta s} \approx \frac{-1}{5/3} = -.6, \quad \frac{\partial w}{\partial y} \bigg|_j \approx \frac{\Delta w}{\Delta s} \approx \frac{-1}{1} = -1. \)

e) If \( u \) is the direction of \( i + j \), we have \( \frac{dw}{ds} \bigg|_u \approx \frac{\Delta w}{\Delta s} \approx \frac{1}{.5} = 2 \)

f) If \( u \) is the direction of \( i - j \), we have \( \frac{dw}{ds} \bigg|_u \approx \frac{\Delta w}{\Delta s} \approx \frac{-1}{5/4} = -8. \)

g) The gradient is 0 at a local extremum point: here at the point marked giving the location of the hilltop.
2E. Chain Rule

2E-1

a) (i) \( \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = yz \cdot 1 + xz \cdot 2t + xy \cdot 3t^2 = t^5 + 2t^5 + 3t^5 = 6t^5 \)

(ii) \( w = xyz = t^6; \quad \frac{dw}{dt} = 6t^5 \)

b) (i) \( \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = 2x(-\sin t) - 2y(\cos t) = -4 \sin t \cos t \)

(ii) \( w = x^2 - y^2 = \cos^2 t - \sin^2 t = \cos 2t; \quad \frac{dw}{dt} = -2 \sin 2t \)

c) (i) \( \frac{dw}{dt} = \frac{2u}{u^2 + v^2} (-2 \sin t) + \frac{2v}{u^2 + v^2} (2 \cos t) = -\cos t \sin t + \sin t \cos t = 0 \)

(ii) \( w = \ln(u^2 + v^2) = \ln(4 \cos^2 t + 4 \sin^2 t) = \ln 4; \quad \frac{dw}{dt} = 0. \)

2E-2

a) The value \( t = 0 \) corresponds to the point \((x(0), y(0)) = (1, 0) = P. \)

\[ \frac{dw}{dt}\bigg|_0 = \frac{\partial w}{\partial x}\bigg|_0 \frac{dx}{dt}\bigg|_0 + \frac{\partial w}{\partial y}\bigg|_0 \frac{dy}{dt}\bigg|_0 = -2 \sin t + 3 \cos t \bigg|_0 = 3. \]

b) \( \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = y(-\sin t) + x(\cos t) = -\sin^2 t + \cos^2 t = \cos 2t. \)

\( \frac{dw}{dt} = 0 \) when \( 2t = \frac{\pi}{2} + n\pi, \) therefore when \( t = \frac{\pi}{4} + \frac{n\pi}{2}. \)

c) \( t = 1 \) corresponds to the point \((x(1), y(1), z(1)) = (1, 1, 1). \)

\[ \frac{df}{dt}\bigg|_1 = 1 \cdot \frac{dx}{dt}\bigg|_1 + 1 \cdot \frac{dy}{dt}\bigg|_1 + 2 \cdot \frac{dz}{dt}\bigg|_1 = 1 \cdot 1 - 1 \cdot 2 + 2 \cdot 3 = 5. \]

d) \( \frac{df}{dt} = 3x^2y \frac{dx}{dt} + (x^3 + z) \frac{dy}{dt} + y \frac{dz}{dt} = 3t^4 \cdot 1 + 2x^3 \cdot 2t + t^2 \cdot 3t^2 = 10t^4. \)

2E-3

a) Let \( w = uv, \) where \( u = u(t), \ v = v(t); \)

\[ \frac{dw}{dt} = \frac{\partial w}{\partial u} \frac{du}{dt} + \frac{\partial w}{\partial v} \frac{dv}{dt} = \frac{du}{dt} + u \frac{dv}{dt}. \]

b) \( \frac{d(uvw)}{dt} = vw \frac{du}{dt} + uw \frac{dv}{dt} + uv \frac{dw}{dt}; \quad e^{2t} \sin t + 2e^{2t} \sin t + te^{2t} \cos t. \)

2E-4

The values \( u = 1, \ v = 1 \) correspond to the point \( x = 0, y = 1. \) At this point,

\[ \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} = 2 \cdot 2u + 3 \cdot v = 2 \cdot 2 + 3 = 7. \]

\[ \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} = 2 \cdot (-2v) + 3 \cdot u = 2 \cdot (-2) + 3 \cdot 1 = -1. \]

2E-5

a) \( w_r = w_x x_r + w_y y_r = w_x \cos \theta + w_y \sin \theta \)

\( w_\theta = w_x x_\theta + w_y y_\theta = w_x (-r \sin \theta) + w_y (r \cos \theta) \)

Therefore,

\[ (w_r)^2 + (w_\theta)^2 \]

\[ = (w_x)^2 (\cos^2 \theta + \sin^2 \theta) + (w_y)^2 (\sin^2 \theta + \cos^2 \theta) + 2w_x w_y \cos \theta \sin \theta - 2w_x w_y \sin \theta \cos \theta \]

\[ = (w_x)^2 + (w_y)^2. \]
b) The point \( r = \sqrt{2}, \theta = \pi/4 \) in polar coordinates corresponds to the point \( x = 1, y = 1 \). Using the chain rule equations in part (a),
\[
\begin{align*}
w_r &= w_x \cos \theta + w_y \sin \theta; \\
w_\theta &= w_x (-r \sin \theta) + w_y (r \cos \theta)
\end{align*}
\]
but evaluating all the partial derivatives at the point, we get
\[
\begin{align*}
w_r &= 2 \cdot \frac{1}{2} \sqrt{2} - 1 \cdot \frac{1}{2} \sqrt{2} = \frac{1}{2} \sqrt{2}; \\
w_\theta &= 2(-\frac{1}{2}) \sqrt{2} - \frac{1}{2} \sqrt{2} = -\frac{3}{2} \sqrt{2}; \\
\end{align*}
\]
\[
(w_r)^2 + \left(\frac{1}{r} (w_\theta)\right)^2 = \frac{1}{2} + \frac{9}{2} = 5; \\
(w_\theta)^2 + (w_\theta)^2 = 2^2 + (-1)^2 = 5.
\]

2E-6 \( w_u = w_x \cdot 2u + w_y \cdot 2v; \quad w_v = w_x \cdot (-2v) + w_y \cdot 2u, \) by the chain rule.
Therefore
\[
\begin{align*}
(w_u)^2 + (w_v)^2 &= [4u^2(w_x) + 4v^2(w_y)^2 + 4uvw_z w_y] + [4v^2(w_x) + 4u^2(w_y)^2 - 4uvw_x w_y]
\end{align*}
\]
\[
= 4(u^2 + v^2)\left[(w_x)^2 + (w_y)^2\right].
\]

2E-7 By the chain rule, \( f_u = f_x x_u + f_y y_u, \quad f_v = f_x x_v + f_y y_v; \) therefore
\[
(f_u f_v) = (f_x f_y) \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}
\]

2E-8 a) By the chain rule for functions of one variable,
\[
\frac{\partial w}{\partial x} = f'(u) \cdot \frac{\partial u}{\partial x} = f'(u) \cdot \frac{-y}{x^2}; \\
\frac{\partial w}{\partial y} = f'(u) \cdot \frac{\partial u}{\partial y} = f'(u) \cdot \frac{1}{x};
\]
Therefore,
\[
x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} = f'(u) \cdot \frac{-y}{x} + f'(u) \cdot \frac{y}{x} = 0.
\]

2F. Maximum-minimum Problems

2F-1 In these, denote by \( D = x^2 + y^2 + z^2 \) the square of the distance from the point \((x, y, z)\) to the origin; then the point which minimizes \( D \) will also minimize the actual distance.

a) Since \( z^2 = \frac{1}{xy} \), we get on substituting, \( D = x^2 + y^2 + \frac{1}{xy} \) with \( x \) and \( y \) independent; setting the partial derivatives equal to zero, we get
\[
D_x = 2x - \frac{1}{x^2 y} = 0; \quad D_y = 2y - \frac{1}{y^2 x} = 0; \quad \text{or} \quad 2x^2 = \frac{1}{xy}, \quad 2y^2 = \frac{1}{xy}.
\]
Solving, we see first that \( x^2 = \frac{1}{2xy} = y^2 \), from which \( y = \pm x \).

If \( y = x \), then \( x^4 = \frac{1}{2} \) and \( x = y = 2^{-1/4} \), and so \( z = 2^{1/4} \); if \( y = -x \), then \( x^4 = -\frac{1}{2} \) and there are no solutions. Thus the unique point is \((1/2^{1/4}, 1/2^{1/4}, 2^{1/4})\).

b) Using the relation \( x^2 = 1 + yz \) to eliminate \( x \), we have \( D = 1 + yz + y^2 + z^2 \), with \( y \) and \( z \) independent; setting the partial derivatives equal to zero, we get
\[
D_y = 2y + z = 0, \quad D_z = 2z + y = 0;
\]
solving, these equations only have the solution \( y = z = 0 \); therefore \( x = \pm 1 \), and there are two points: \((\pm1, 0, 0)\), both at distance 1 from the origin.

2F-2 Letting \( x \) be the length of the ends, \( y \) the length of the sides, and \( z \) the height, we have
\[
\text{total area of cardboard} \quad A = 3xy + 4xz + 2yz, \quad \text{volume} \quad V = xyz = 1.
\]
Eliminating \( z \) to make the remaining variables independent, and equating the partials to zero, we get
\[ A = 3xy + \frac{4}{y} + \frac{2}{x} \quad \text{and} \quad A_x = 3y - \frac{2}{x^2} = 0, \quad A_y = 3x - \frac{4}{y^2} = 0. \]

From these last two equations, we get
\[ 3xy = \frac{2}{x}, \quad 3xy = \frac{4}{y} \quad \Rightarrow \quad \frac{2}{x} = \frac{4}{y} \quad \Rightarrow \quad y = 2x. \]

Therefore the proportions of the most economical box are \( x : y : z = 1 : 2 : \frac{3}{2} \).

2F-5 The cost is \( C = xy + xz + 4yz + 4xz \), where the successive terms represent in turn the bottom, back, two sides, and front; i.e., the problem is:

\[ \text{minimize: } C = xy + 5xz + 4yz, \quad \text{with the constraint: } xyz = V = 2.5. \]

Substituting \( z = V/xy \) into \( C \), we get
\[ C = xy + \frac{5V}{y} + \frac{4V}{x}; \quad \partial C/\partial x = y - \frac{4V}{x^2}, \quad \partial C/\partial y = x - \frac{5V}{y^2}. \]

We set the two partial derivatives equal to zero and solving the resulting equations simultaneously, by eliminating \( y \); we get \( x^3 = \frac{16V}{5} = 8 \), (using \( V = 5/2 \)), so \( x = 2, \ y = \frac{5}{2}, \ z = \frac{1}{2} \).

2G. Least-squares Interpolation

2G-1 Find \( y = mx + b \) that best fits \((1, 1), \ (2, 3), \ (3, 2)\).

\[ D = (m + b - 1)^2 + (2m + b - 3)^2 + (3m + b - 2)^2 \]
\[ \partial D/\partial m = 2(m + b - 1) + 4(2m + b - 3) + 6(3m + b - 2) = 2(14m + 6b - 13) \]
\[ \partial D/\partial b = 2(m + b - 1) + 2(2m + b - 3) + 2(3m + b - 2) = 2(6m + 3b - 6). \]

Thus the equations \( \partial D/\partial m = 0 \) and \( \partial D/\partial b = 0 \) are \( \begin{cases} 14m + 6b = 13 \quad \text{and} \quad 6m + 3b = 6 \end{cases} \), whose solution is \( m = \frac{1}{2}, \ b = 1 \), and the line is \( y = \frac{1}{2}x + 1 \).

2G-4 \( D = \sum_i (a + bx_i + cy_i - z_i)^2 \). The equations are
\[ \partial D/\partial a = \sum 2(a + bx_i + cy_i - z_i) = 0 \]
\[ \partial D/\partial b = \sum 2x_i(a + bx_i + cy_i - z_i) = 0 \]
\[ \partial D/\partial c = \sum 2y_i(a + bx_i + cy_i - z_i) = 0 \]

Cancel the 2's; the equations become (on the right, \( x = [x_1, \ldots, x_n], \ 1 = [1, \ldots, 1] \), etc.)
\[ na + (\sum x_i)b + (\sum y_i)c = \sum z_i \quad \text{or} \quad (x \cdot 1) a + (x \cdot x) b + (x \cdot y) c = z \cdot 1 \]
\[ (\sum x_i)a + (\sum x_i^2)b + (\sum x_iy_i)c = \sum x_iz_i \quad \text{or} \quad (x \cdot 1) a + (x \cdot x) b + (x \cdot y) c = x \cdot z \]
\[ (\sum y_i)a + (\sum x_iy_i)b + (\sum y_i^2)c = \sum y_iz_i \quad \text{or} \quad (y \cdot 1) a + (x \cdot x) b + (y \cdot y) c = y \cdot z \]
2H. Max-min: 2nd Derivative Criterion; Boundary Curves

2H-1

a) \( f_x = 0 : \ 2x - y = 3; \quad f_y = 0 : \ -x - 4y = 3 \)
   \[ A = f_{xx} = 2; \quad B = f_{xy} = -1; \quad C = f_{yy} = -4; \quad AC - B^2 = -9 < 0; \text{ saddle point} \]

b) \( f_x = 0 : \ 6x + y = 1; \quad f_y = 0 : \ x + 2y = 2 \)
   \[ A = f_{xx} = 6; \quad B = f_{xy} = 1; \quad C = f_{yy} = 2; \quad AC - B^2 = 11 > 0; \text{ local minimum} \]

c) \( f_x = 0 : \ 8x^3 - y = 0; \quad f_y = 0 : \ 2y - x = 0 \)
   Eliminating \( y \), we get
   \[ 16x^3 - x = 0, \text{ or } x(16x^2 - 1) = 0 \Rightarrow x = 0, \ x = \frac{1}{4}, \ x = -\frac{1}{4}, \text{ giving the critical points} \]
   \[ (0,0), \quad \left( \frac{1}{4}, \frac{1}{8} \right), \quad \left( -\frac{1}{4}, -\frac{1}{8} \right). \]
   Since \( f_{xx} = 24x^2 \), \( f_{xy} = -1 \), \( f_{yy} = 2 \), we get for the three points respectively:
   \[ (0,0) : \Delta = -1 \text{ (saddle)}; \quad \left( \frac{1}{4}, \frac{1}{8} \right) : \Delta = 2 \text{ (minimum)}; \quad \left( -\frac{1}{4}, -\frac{1}{8} \right) : \Delta = 2 \text{ (minimum)} \]

d) \( f_x = 0 : \ 3x^2 - 3y = 0; \quad f_y = 0 : \ -3x + 3y^2 = 0 \).
   Eliminating \( y \) gives
   \[ -x + x^4 = 0, \text{ or } x(x^3 - 1) = 0 \Rightarrow x = 0, \ y = 0 \text{ or } x = 1, \ y = 1. \]
   Since \( f_{xx} = 6x \), \( f_{xy} = -3 \), \( f_{yy} = 6y \), we get for the two critical points respectively:
   \[ (0,0) : \ AC - B^2 = -9 \text{ (saddle)}; \quad (1,1) : \ AC - B^2 = 27 \text{ (minimum)} \]

e) \( f_x = 0 : \ 3x^2(y^3 + 1) = 0; \quad f_y = 0 : \ 3y^2(x^3 + 1) = 0 \);
   solving simultaneously, we get from the first equation that either \( x = 0 \) or \( y = -1 \); finding in each case the other coordinate then leads to the two critical points \((0,0)\) and \((-1,-1)\).
   Since \( f_{xx} = 6x(y^3 + 1) \), \( f_{xy} = 3x^2 - 3y^2 \), \( f_{yy} = 6y(x^3 + 1) \), we have
   \[ (-1,-1) : \ AC - B^2 = -9 \text{ (saddle)}; \quad (0,0) : \ AC - B^2 = 0, \text{ test fails.} \]

   (By studying the behavior of \( f(x, y) \) on the lines \( y = mx \), for different values of \( m \), it is possible to see that also \((0,0)\) is a saddle point.)

2H-3

The region \( R \) has no critical points; namely, the equations \( f_x = 0 \) and \( f_y = 0 \) are

\[ 2x + 2 = 0, \quad 2y + 4 = 0 \Rightarrow x = -1, \ y = -2, \]

but this point is not in \( R \). We therefore investigate the diagonal boundary of \( R \), using the parametrization \( x = t, \ y = -t \). Restricted to this line, \( f(x, y) \) becomes a function of \( t \) alone, which we denote by \( g(t) \), and we look for its maxima and minima.

\[ g(t) = f(t, -t) = 2t^2 - 2t - 1; \quad g'(t) = 4t - 2, \text{ which is 0 at } t = 1/2. \]

This point is evidently a minimum for \( g(t) \); there is no maximum: \( g(t) \) tends to \( \infty \). Therefore for \( f(x, y) \) on \( R \), the minimum occurs at the point \((1/2, -1/2)\), and there is no maximum; \( f(x, y) \) tends to infinity in different directions in \( R \).
2H-4  We have \( f_x = y - 1, \ f_y = x - 1 \), so the only critical point is at \((1, 1)\).

a)  On the two sides of the boundary, the function \( f(x, y) \) becomes respectively

\[
y = 0: \quad f(x, y) = -x + 2; \quad x = 0: \quad f(x, y) = -y + 2.
\]

Since the function is linear and decreasing on both sides, it has no minimum points (informally, the minimum is \(-\infty\)). Since \( f(1, 1) = 1 \) and \( f(x, x) = x^2 - 2x + 2 \to \infty \) as \( x \to \infty \), the maximum of \( f \) on the first quadrant is \( \infty \).

b)  Continuing the reasoning of (a) to find the maximum and minimum points of \( f(x, y) \) on the boundary, on the other two sides of the boundary square, the function \( f(x, y) \) becomes

\[
y = 2: \quad f(x, y) = x \quad \text{and} \quad x = 2: \quad f(x, y) = y.
\]

Since \( f(x, y) \) is thus increasing or decreasing on each of the four sides, the maximum and minimum points on the boundary square \( R \) can only occur at the four corner points; evaluating \( f(x, y) \) at these four points, we find

\[
f(0, 0) = 2; \quad f(2, 2) = 2; \quad f(2, 0) = 0; \quad f(0, 2) = 0.
\]

As in (a), since \( f(1, 1) = 1 \),

maximum points of \( f \) on \( R \): \((0, 0)\) and \((2, 2)\); minimum points: \((2, 0)\) and \((0, 2)\).

c)  The data indicates that \((1, 1)\) is probably a saddle point. Confirming this, we have \( f_{xx} = 0, \ f_{xy} = 1, \ f_{yy} = 0 \) for all \( x \) and \( y \); therefore \( AC - B^2 = -1 < 0 \), so \((1, 1)\) is a saddle point, by the 2nd-derivative criterion.

2H-5  Since \( f(x, y) \) is linear, it will not have critical points: namely, for all \( x \) and \( y \) we have \( f_x = 1, \ f_y = \sqrt{3} \). So any maxima or minima must occur on the boundary circle.

We parametrize the circle by \( x = \cos \theta, \ y = \sin \theta \); restricted to this boundary circle, \( f(x, y) \) becomes a function of \( \theta \) alone which we call \( g(\theta) \):

\[
g(\theta) = f(\cos \theta, \sin \theta) = \cos \theta + \sqrt{3} \sin \theta + 2.
\]

Proceeding in the usual way to find the maxima and minima of \( g(\theta) \), we get

\[
g'(\theta) = -\sin \theta + \sqrt{3} \cos \theta = 0, \quad \text{or} \quad \tan \theta = \sqrt{3}.
\]

It follows that the two critical points of \( g(\theta) \) are \( \theta = \frac{\pi}{3} \) and \( \frac{4\pi}{3} \); evaluating \( g \) at these two points, we get \( g(\pi/3) = 4 \) (the maximum), and \( g(4\pi/3) = 0 \) (the minimum).

Thus the maximum of \( f(x, y) \) in the circular disc \( R \) is at \((1/2, \sqrt{3}/2)\), while the minimum is at \((-1/2, -\sqrt{3}/2)\).

2H-6  a)  Since \( z = 4 - x - y \), the problem is to find on \( R \) the maximum and minimum of the total area

\[
f(x, y) = xy + \frac{1}{4}(4 - x - y)^2
\]

where \( R \) is the triangle given by \( R: \ 0 \leq x, \ 0 \leq y, \ x + y \leq 4 \).

To find the critical points of \( f(x, y) \), the equations \( f_x = 0 \) and \( f_y = 0 \) are respectively

\[
y - \frac{1}{2}(4 - x - y) = 0; \quad x - \frac{1}{2}(4 - x - y) = 0,
\]

which imply first that \( x = y \), and from this, \( x - \frac{1}{2}(4 - 2x) \); the unique solution is \( x = 1, \ y = 1 \).
The region $R$ is a triangle, on whose sides $f(x, y)$ takes respectively the values

\begin{align*}
\text{bottom: } y = 0; & \quad f = \frac{1}{4}(4 - x)^2; \\
\text{left side: } x = 0; & \quad f = \frac{1}{4}(4 - y)^2; \\
\text{diagonal } y = 4 - x; & \quad f = x(4 - x).
\end{align*}

On the bottom and side, $f$ is decreasing; on the diagonal, $f$ has a maximum at $x = 2, y = 2$. Therefore we need to examine the three corner points and $(2, 2)$ as candidates for maximum and minimum points, as well as the critical point $(1, 1)$. We find

\[
f(0, 0) = 4; \quad f(4, 0) = 0; \quad f(0, 4) = 0; \quad f(2, 2) = 4 \quad f(1, 1) = 2.
\]

It follows that the critical point is just a saddle point; to get the maximum total area 4, make $x = y = 0, z = 4$, or $x = y = 2, z = 0$, either of which gives a point “rectangle” and a square of side 2; for the minimum total area 0, take for example $x = 0, y = 4, z = 0$, which gives a “rectangle” of length 4 with zero area, and a point square.

b) We have $f_{xx} = \frac{1}{2}, f_{xy} = \frac{3}{2}, f_{yy} = \frac{1}{2}$ for all $x$ and $y$; therefore $AC - B^2 = -2 < 0$, so $(1, 1)$ is a saddle point, by the 2nd-derivative criterion.

2H-7 a) $f_x = 4x - 2y - 2, f_y = -2x + 2y$; setting these equals 0 and solving simultaneously, we get $x = 1, y = 1$, which is therefore the only critical point.

By one-variable calculus, $f(x, y)$ is increasing on the bottom and decreasing on the right side; on the left side it has a minimum at $(0, 0)$, and on the top a minimum at $(\frac{3}{2}, 2)$. Thus the maximum and minimum points on the boundary rectangle $R$ can only occur at the four corner points, or at $(0, 0)$ or $(\frac{3}{2}, 2)$. At these we find:

\[
f(0, -1) = 1; \quad f(0, 2) = 4; \quad f(2, -1) = 9; \quad f(2, 2) = 0; \quad f(\frac{3}{2}, 2) = -\frac{1}{2}; \quad f(0, 0) = 0.
\]

At the critical point $f(1, 1) = -1$; comparing with the above, it is a minimum; therefore, maximum point of $f(x, y)$ on $R$: $(2, -1)$ minimum point of $f(x, y)$ on $R$: $(1, 1)$

b) We have $f_{xx} = 4, f_{xy} = -2, f_{yy} = 2$ for all $x$ and $y$; therefore $AC - B^2 = 4 > 0$ and $A = 4 > 0$, so $(1, 1)$ is a minimum point, by the 2nd-derivative criterion.

2I. Lagrange Multipliers

2I-1 Letting $P : (x, y, z)$ be the point, in both problems we want to maximize $V = xyz$, subject to a constraint $f(x, y, z) = c$. The Lagrange equations for this, in vector form, are

\[
\nabla (xyz) = \lambda \cdot \nabla f(x, y, z), \quad f(x, y, z) = c.
\]

a) Here $f = c$ is $x + 2y + 3z = 18$; equating components, the Lagrange equations become

\[
yz = \lambda, \quad xz = 2\lambda, \quad xy = 3\lambda; \quad x + 2y + 3z = 18.
\]

To solve these symmetrically, multiply the left sides respectively by $x$, $y$, and $z$ to make them equal; this gives

\[
\lambda x = 2\lambda y = 3\lambda z, \quad \text{or} \quad x = 2y = 3z = 6, \text{ since the sum is 18}
\]
We get therefore as the answer \( x = 6, \ y = 3, \ z = 2 \). This is a maximum point, since if \( P \) lies on the triangular boundary of the region in the first octant over which it varies, the volume of the box is zero.

b) Here \( f = c \) is \( x^2 + 2y^2 + 4z^2 = 12 \); equating components, the Lagrange equations become

\[
yz = \lambda \cdot 2x, \quad xz = \lambda \cdot 4y, \quad xy = \lambda \cdot 8z; \quad x^2 + 2y^2 + 4z^2 = 12.
\]

To solve these symmetrically, multiply the left sides respectively by \( x, y, z \) and make them equal; this gives

\[
\lambda \cdot 2x^2 = \lambda \cdot 4y^2 = \lambda \cdot 8z^2, \quad or \quad x^2 = 2y^2 = 4z^2 = 4, \quad since \ the \ sum \ is \ 12.
\]

We get therefore as the answer \( x = 2, \ y = \sqrt{2}, \ z = 1 \). This is a maximum point, since if \( P \) lies on the boundary of the region in the first octant over which it varies (1/8 of the ellipsoid), the volume of the box is zero.

2I-2 Since we want to minimize \( x^2 + y^2 + z^2 \), subject to the constraint \( x^2y^2z = 6\sqrt{3} \), the Lagrange multiplier equations are

\[
2x = \lambda \cdot 3x^2y^2z, \quad 2y = \lambda \cdot 2x^3yz, \quad 2z = \lambda \cdot x^3y^2; \quad x^2y^2z = 6\sqrt{3}.
\]

To solve them symmetrically, multiply the first three equations respectively by \( x, y, z \), then divide them through respectively by 3, 2, and 1; this makes the right sides equal, so that, after canceling 2 from every numerator, we get

\[
\frac{x^2}{3} = \frac{y^2}{2} = \frac{z^2}{3}; \quad therefore \quad x = z\sqrt{3}, \quad y = z\sqrt{2}.
\]

Substituting into \( x^2y^2z = 6\sqrt{3} \), we get \( 3\sqrt{3}x^3 \cdot 2z^2 \cdot z = 6\sqrt{3} \), which gives as the answer, \( x = \sqrt{3}, \ y = \sqrt{2}, \ z = 1 \).

This is clearly a minimum, since if \( P \) is near one of the coordinate planes, one of the variables is close to zero and therefore one of the others must be large, since \( x^2y^2z = 6\sqrt{3} \); thus \( P \) will be far from the origin.

2I-3 Referring to the solution of 2F-2, we let \( x \) be the length of the ends, \( y \) the length of the sides, and \( z \) the height, and get

\[
\text{total area of cardboard } \ A = 3xy + 4xz + 2yz, \quad \text{volume } \ V = xyz = 1.
\]

The Lagrange multiplier equations \( \nabla A = \lambda \cdot \nabla (xyz) \); \( xyz = 1 \), then become

\[
3y + 4z = \lambda yz, \quad 3x + 2z = \lambda xz, \quad 4x + 2y = \lambda xy, \quad xyz = 1.
\]

To solve these equations for \( x, y, z, \lambda \), treat them symmetrically. Divide the first equation through by \( yz \), and treat the next two equations analogously, to get

\[
3/z + 4/y = \lambda, \quad 3/z + 2/x = \lambda, \quad 4/y + 2/x = \lambda,
\]

which by subtracting the equations in pairs leads to \( 3/z = 4/y = 2/x \); setting these all equal to \( k \), we get \( x = 2/k, y = 4/k, z = 3/k \), which shows the proportions using least cardboard are \( x : y : z = 2 : 4 : 3 \).

To find the actual values of \( x, y, \) and \( z \), we set \( 1/k = m \); then substituting into \( xyz = 1 \) gives \((2m)(4m)(3m) = 1\), from which \( m^3 = 1/24, m = 1/2 \cdot 3^{1/3}, \) giving finally

\[
x = \frac{1}{3^{1/3}}, \quad y = \frac{2}{3^{1/3}}, \quad z = \frac{3}{2 \cdot 3^{1/3}}.
\]
The equations for the cost $C$ and the volume $V$ are $xy + 4yz + 6xz = C$ and $xyz = V$. The Lagrange multiplier equations for the two problems are

a) $yz = \lambda(y + 6z)$, $xz = \lambda(x + 4z)$, $xy = \lambda(4y + 6x)$; $xy + 4yz + 6xz = 72$

b) $y + 6z = \mu \cdot yz$, $x + 4z = \mu \cdot xz$, $4y + 6x = \mu \cdot xy$; $xyz = 24$

The first three equations are the same in both cases, since we can set $\mu = 1/\lambda$. Solving the first three equations in (a) symmetrically, we multiply the equations through by $x$, $y$, and $z$ respectively, which makes the left sides equal; since the right sides are therefore equal, we get after canceling the $\lambda$,

$$xy + 6xz = xy + 4yz = 4yz + 6xz,$$

which implies $xy = 4yz = 6xz$.

a) Since the sum of the three equal products is 72, by hypothesis, we get

$$xy = 24, \ yz = 6, \ xz = 4;$$

from the first two we get $x = 4z$, and from the first and third we get $y = 6z$, which lead to the solution $x = 4, y = 6, z = 1$.

b) Dividing $xy = 4yz = 6xz$ by $xyz$ leads after cross-multiplication to $x = 4z$, $y = 6z$; since by hypothesis, $xyz = 24$, again this leads to the solution $x = 4, y = 6, z = 1$.

### 2J. Non-independent Variables

2J-1 a) $(\frac{\partial w}{\partial y})_z$ means that $x$ is the dependent variable; get rid of it by writing

$$w = (z - y)^2 + y^2 + z^2 = z + z^2.$$ This shows that $(\frac{\partial w}{\partial y})_z = 0$.

b) To calculate $(\frac{\partial w}{\partial z})_y$, once again $x$ is the dependent variable; as in part (a), we have $w = z + z^2$ and so $(\frac{\partial w}{\partial z})_y = 1 + 2z$.

2J-2 a) Differentiating $z = x^2 + y^2$ w.r.t. $y$: $0 = 2x (\frac{\partial x}{\partial y})_z + 2y$; so $(\frac{\partial x}{\partial y})_z = -\frac{y}{x}$.

By the chain rule, $(\frac{\partial w}{\partial y})_z = 2x (\frac{\partial x}{\partial y})_z + 2y = 2x \left( -\frac{y}{x} \right) + 2y = 0$.

Differentiating $z = x^2 + y^2$ with respect to $z$: $1 = 2x (\frac{\partial x}{\partial z})_y$; so $(\frac{\partial x}{\partial z})_y = \frac{1}{2x}$.

By the chain rule, $(\frac{\partial w}{\partial z})_y = 2x (\frac{\partial x}{\partial z})_y + 2z = 1 + 2z$.

b) Using differentials, $dw = 2xdx + 2ydy + 2zdz$, $dz = 2xdx + 2ydy$; since the independent variables are $y$ and $z$, we eliminate $dx$ by subtracting the second equation from the first, which gives $dw = 0 dy + (1 + 2z) dz$; therefore we get $(\frac{\partial w}{\partial y})_z = 0$, $(\frac{\partial w}{\partial z})_y = 1 + 2z$. 

2J-3  a) To calculate \( \left( \frac{\partial w}{\partial t} \right)_{x,z} \), we see that \( y \) is the dependent variable; solving for it, we get \( y = \frac{zt}{x} \); using the chain rule, \( \left( \frac{\partial w}{\partial t} \right)_{x,z} = x^3 \left( \frac{\partial y}{\partial t} \right)_{x,z} - z^2 = x^3 \frac{z}{x} - z^2 = x^2 z - z^2 \).

b) Similarly, \( \left( \frac{\partial w}{\partial z} \right)_{x,y} \) means that \( t \) is the dependent variable; since \( t = \frac{xy}{z} \), we have by the chain rule, \( \left( \frac{\partial w}{\partial z} \right)_{x,y} = -2zt - z^2 \left( \frac{\partial t}{\partial z} \right)_{x,y} = -2zt - z^2 \cdot \frac{-xy}{z^2} = -zt \).

2J-4  The differentials are calculated in equation (4).

a) Since \( x, z, t \) are independent, we eliminate \( dy \) by solving the second equation for \( xdy \), substituting this into the first equation, and grouping terms:
\[
dw = 2x^2y \, dx + (x^2z - z^2) \, dt + (x^2t - 2zt) \, dz,
\]
which shows that \( \left( \frac{\partial w}{\partial t} \right)_{x,z} = x^2z - z^2 \).

b) Since \( x, y, z \) are independent, we eliminate \( dt \) by solving the second equation for \( z \, dt \), substituting this into the first equation, and grouping terms:
\[
dw = (3x^2y - zy) \, dx + (x^3 - zx) \, dy - zt \, dz,
\]
which shows that \( \left( \frac{\partial w}{\partial z} \right)_{x,y} = -zt \).

2J-5  a) If \( pv = nRT \), then \( \left( \frac{\partial S}{\partial p} \right)_v = S_p + S_T \cdot \left( \frac{\partial T}{\partial p} \right)_v = S_p + S_T \cdot \frac{v}{nR} \).

b) Similarly, we have \( \left( \frac{\partial S}{\partial T} \right)_v = S_T + S_p \cdot \left( \frac{\partial p}{\partial T} \right)_v = S_T + S_p \cdot \frac{nR}{v} \).

2J-6  a) \( \left( \frac{\partial w}{\partial u} \right)_x = 3u^2 - v^2 - u \cdot 2v \left( \frac{\partial v}{\partial u} \right)_x = 3u^2 - v^2 - 2uv \).

\( \left( \frac{\partial w}{\partial x} \right)_u = -u \cdot 2v \left( \frac{\partial v}{\partial x} \right)_u = -2uv \).

b) \( dw = (3u^2 - v^2) \, du - 2uv \, dv \); \( du = x \, dy + y \, dx \); \( dv = du + dx \);
for both derivatives, \( u \) and \( x \) are the independent variables, so we eliminate \( dv \), getting
\[
dw = (3u^2 - v^2) \, du - 2uv(du + dx) = (3u^2 - v^2 - 2uv) \, du - 2uv \, dx,
\]
whose coefficients are \( \left( \frac{\partial w}{\partial u} \right)_x \) and \( \left( \frac{\partial w}{\partial x} \right)_u \).

2J-7  Since we need both derivatives for the gradient, we use differentials.
\[
df = 2dx + dy - 3dz \quad \text{at } P; \quad dz = 2x \, dx + dy = 2 \, dx + dy \quad \text{at } P;
\]
the independent variables are to be \( x \) and \( z \), so we eliminate \( dy \), getting
\[
df = 0 \, dx - 2 \, dz \quad \text{at the point } (x, z) = (1, 1). \quad \text{So } \nabla g = (0, -2) \quad \text{at } (1, 1).
\]

2J-8  To calculate \( \left( \frac{\partial w}{\partial r} \right)_\theta \), note that \( w = r \sin \theta \). Therefore, \( \left( \frac{\partial w}{\partial r} \right)_\theta = |\sin \theta| \).
2K. Partial Differential Equations

2K-1  \( w = \frac{1}{2} \ln(x^2 + y^2) \). If \((x, y) \neq (0, 0)\), then
\[
\begin{align*}
    w_{xx} &= \frac{\partial}{\partial x}(w_x) = \frac{\partial}{\partial x} \left( \frac{x}{x^2 + y^2} \right) = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \\
    w_{yy} &= \frac{\partial}{\partial y}(w_y) = \frac{\partial}{\partial y} \left( \frac{y}{x^2 + y^2} \right) = \frac{x^2 - y^2}{(x^2 + y^2)^2},
\end{align*}
\]
Therefore \( w \) satisfies the two-dimensional Laplace equation, \( w_{xx} + w_{yy} = 0 \); we exclude the point \((0,0)\) since \( \ln 0 \) is not defined.

2K-2  If \( w = (x^2 + y^2 + z^2)^n \), then \( \frac{\partial}{\partial x}(w_x) = \frac{\partial}{\partial x} (2x \cdot n(x^2 + y^2 + z^2)^{n-1}) \)
\[
= 2n(x^2 + y^2 + z^2)^{n-1} + 4x^2 n(n-1)(x^2 + y^2 + z^2)^{n-2}
\]
We get \( w_{yy} \) and \( w_{zz} \) by symmetry; adding and combining, we get
\[
w_{xx} + w_{yy} + w_{zz} = 6n(x^2 + y^2 + z^2)^{n-1} + 4(x^2 + y^2 + z^2)n(n-1)(x^2 + y^2 + z^2)^{n-2}
\]
\[
= 2n(2n+1)(x^2 + y^2 + z^2)^{n-1}, \quad \text{which is identically zero if } n = 0, \text{ or if } n = -1/2.
\]

2K-3  a) \( w = ax^2 + bxy + cy^2; \quad w_{xx} = 2a, \quad w_{yy} = 2c. \)
\[
w_{xx} + w_{yy} = 0 \quad \Rightarrow \quad 2a + 2c = 0, \quad \text{or } c = -a.
\]
Therefore all quadratic polynomials satisfying the Laplace equation are of the form
\[
a(x^2 + bxy - ay^2) = a(x^2 - y^2) + bxy;
\]
i.e., linear combinations of the two polynomials \( f(x,y) = x^2 - y^2 \) and \( g(x,y) = xy \).

2K-4  The one-dimensional wave equation is \( w_{xx} = \frac{1}{c^2} w_{tt} \). So
\[
w = f(x + ct) + g(x - ct) \quad \Rightarrow \quad w_{xx} = f''(x + ct) + g''(x - ct)
\]
\[
\Rightarrow \quad w_{tt} = c^2 f''(x + ct) - c^2 g''(x - ct),
\]
\[
\Rightarrow \quad w_{tt} = c^2 f''(x + ct) + c^2 g''(x - ct) = c^2 w_{xx},
\]
which shows \( w \) satisfies the wave equation.

2K-5  The one-dimensional heat equation is \( w_{xx} = \frac{1}{\alpha^2} w_t \). So if \( w(x,t) = \sin kx e^{rt} \), then
\[
w_{xx} = e^{rt} \cdot k^2 (- \sin kx) = -k^2 w.
\]
\[
w_t = re^{rt} \sin kx = r w.
\]
Therefore, we must have \(-k^2 w = \frac{1}{\alpha^2} r w, \) or \( r = -\alpha^2 k^2. \)

However, from the additional condition that \( w = 0 \) at \( x = 1 \), we must have
\[
\sin k e^{rt} = 0;
\]
Therefore \( \sin k = 0 \), and so \( k = n\pi, \) where \( n \) is an integer.

To see what happens to \( w \) as \( t \to \infty \), we note that since \( |\sin kx| \leq 1 \),
\[
|w| = e^{rt} |\sin kx| \leq e^{rt}.
\]
Now, if \( k \neq 0 \), then \( r = -\alpha^2 k^2 \) is negative and \( e^{rt} \to 0 \) as \( t \to \infty \); therefore \( |w| \to 0 \).

Thus \( w \) will be a solution satisfying the given side conditions if \( k = n\pi, \) where \( n \) is a non-zero integer, and \( r = -\alpha^2 k^2. \)
18.02 Notes and Exercises by A. Mattuck with the assistance of T. Shifrin and S. LeDuc, and including a section on non-independent variables by Bjorn Poonen.

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