Lecture 4

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Macdonald

\[ \rightarrow \]

Schur

\[ \rightarrow \]

GUE
GUE corner process

GUE M: $N \times N$ Hermitian with complex off diag Gaussian, real on diag Gaussian.

Call $\lambda^{(n)} = (\lambda_1^{(n)} \leq \cdots \leq \lambda_{N}^{(n)}) = \text{eig}(M)$

$\lambda^{(k)} = (\lambda_1^{(k)} \leq \cdots \leq \lambda_{k}^{(k)}) = \text{eig} [ M^{\text{corner}} ]$

$\lambda$ satisfy Weyl inequalities:

$$\lambda_{N}^{(n)} \geq \lambda_{N-1}^{(n)} \geq \cdots \geq \lambda_{1}^{(n)}$$

Vandermonde det.

$$P(\lambda^{(n)}) = z^{-1} \cdot \frac{V(\lambda^{(n)})^2}{e^{\sum (\lambda_{i}^{(n)})^2}}$$

$$V(\lambda) = \prod_{i < j} (\lambda_i - \lambda_j)$$

Weyl denominator formula

$$P(\lambda^{(n)}, \ldots, \lambda^{(n)}) = \prod_{i} P(\lambda^{(n)}) \cdot \frac{1}{V(\lambda^{(n)})}$$

$$\text{HW: } \int \prod_{i} \lambda_i^{(n)} \ldots \lambda_i^{(n-1)} d\lambda_1^{(n)} \ldots d\lambda_{N-1}^{(n-1)} = 1/V(\lambda^{(n)})$$
1. Compute correlation functions (using $\det = V$) in terms of minors of correlation kernel (Dyson).

Determinantal point process $\Rightarrow$ relatively easy asymptotics.

2. Introduce dynamics on triangular array which preserves corners process/GUE\# (up to scaling).

Dyson's Brownian Motion (DBM):

$$L = V^{-1} \Delta V \quad \Delta \sim \text{Dirichlet Laplacian}$$

$$((LF)(x) = V^{-1}(x) \Delta(V(y)F(y)))_{y=x}.$$ 

DBM pushes GUE measure to scaled (by $t^{1/2}$) version.

Can imagine running DBM for each level, but how to couple them to preserve corners process (and interlacing%)?
Warren's Process \( \text{Not same as eig of evolving GUE matrix} \)

\[ \lambda^{(1)} \sim BM \]

\[ \lambda^{(2)} \sim BM \text{'s reflected on left/right of } \lambda^{(1)} \text{'} \]

so on (reflect off lower particles to stay ordered)

Thm: Push forward of GUE corner-process after-time \( t \) is just (marginally) scaled \( (t^{1/2}) \) version of process.

A discrete time version: (Diaconis-fill / Borodin-Ferrari)

Note \( T^I \lambda = M(\lambda) \cdot \Lambda_{N-1}^{N}(\lambda^{(N)}, \lambda^{(N-1)}) \cdots \Lambda_{1}^{2}(\lambda, \lambda^{(1)}) \)

\( \Lambda^{k} \) \( \lambda \rightarrow \Lambda^{k-1}(\lambda, \lambda^{(k-1)}) := \frac{V(\lambda^{(k-1)})}{V(\lambda^{(k)})} \cdot \Lambda^{k} \text{ Markov} \)

\( \Lambda^{k} \text{ Markov} \text{ (relative volumes)} \) \( \Lambda^{k} \text{ Markov} \)

Assume:

\( \text{All squares commute} \)

\( \Lambda^{k-1} \text{ Markov} \text{ (Markov l.m.s)} \)

\( \Lambda^{k} \text{ Markov} \text{ (Markov l.m.s)} \)
This procedure will survive at Macaulay process level.

\[ \mathcal{M}(\mathcal{N}) = \prod_{n=1}^{\infty} \mathcal{M}(\mathcal{N}) \]

Then if \( P \) is a measure (pass this for reference)

**Thm:** If \( P \) is a measure (pass this for reference)...

Sequential update...

Define \( P : S \to S \) with Mean kernel.
Schur process

Partition: \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq 0) \) \( \lambda_i \in \mathbb{Z}_{\geq 0} \)

\( |\lambda| = \sum \lambda_i \), \( \ell(\lambda) = \# \{ \lambda_i \neq 0 \} \). Eg (4,2,1).

Schur symmetric polynomials in \( x_1, \ldots, x_N = X \)

\[ S_\lambda(X) = \frac{\det(x_i^{\lambda_j+n-j})_{i,j=1}^N}{\det(x_i^{\lambda_j})_{i,j=1}^N} \leftrightarrow V(x) \]

\( \{ S_\lambda : \ell(\lambda) \leq N \} \) form linear basis of sym. poly in \( X \).

Schur measure (Okounkov '01) \( x=(x_1, \ldots, x_N), y=(y_1, \ldots, y_N) \)

\[ P_{x,y}(\lambda) = \frac{S_\lambda(x) S_\lambda(y)}{\Pi(x;y)} \] supported on \( \lambda : \ell(\lambda) \leq N \)

- If \( x_i, y_j \geq 0 \), all \( S_\lambda \)'s and measure is positive

- \( \Pi(x:y) := \sum_\lambda S_\lambda(x) S_\lambda(y) = \prod_{i,j} \frac{1}{1-x_i y_j} \) Cauchy identity
Schur process (Okounkov-Reshetikhin '03)

A measure on $\lambda^{(n)} \geq \lambda^{(n-1)} \geq \ldots \geq \lambda^{(1)}$ partitions

$$P(\lambda^{(n)}, \ldots, \lambda^{(1)}) = \frac{S_{\lambda^{(n)}}(x_1, \ldots, x_n) \cdot S_{\lambda^{(n-1)}}(y_1, \ldots, y_n) \ldots S_{\lambda^{(1)}}(y_1)}{\Pi(x; y)}$$

where

$$S_{\lambda\mu}(u) = u^{|\lambda|-|\mu|} \prod_{\lambda_i > \mu_j} (\lambda_i - \mu_j)$$

Taking all $x_i$'s same and $y_i = 1$, in limit rescale to GUE

Due to determinantal formulas for $S_x$ can see determinantal point process structure on full triangle. (answers question 1 of how to compute)

Projection on $\lambda^{(k)} 

S_{\lambda^{(k)}}(x_1, \ldots, x_n) S_{\lambda^{(k)}}(y_1, \ldots, y_k) \frac{\Pi(x_1, \ldots, x_n; y_1, \ldots, y_k)}{\Pi(x_1, \ldots, x_n; y_1, \ldots, y_k)}$
Question 2: Dynamics (Example 1)

- Process can be written as

\[ m \left( X^{(n)} \right) \Lambda_{n-1}^N \left( X^{(n)}, X^{(n-1)} \right) \ldots \Lambda_1^2 \left( X^{(2)}, X^{(1)} \right) \]

with

\[ m \left( X^{(n)} \right) := \frac{S_{X^{(n)}}(X) S_{X^{(n)}}(Y)}{\Pi(X;Y)} \]

\[ \Lambda_{k-1}^k (\lambda, \mu) := \frac{S_{\mu}(Y_1, \ldots, Y_{k-1}) S_{X^{(k)}}(Y_k)}{S_{\lambda}(Y_1, \ldots, Y_k)} \]

Define Markov kernel

\[ \Lambda_{k-1}^k (\lambda, \mu) \text{ is markov due to "Branching rule"} \]

\[ S_{X}(X_1, \ldots, X_N) = \sum_{\mu \leq \lambda} S_{\mu}(X_1, \ldots, X_{N-1}) \cdot S_{X^{(N)}}(X_N) \]
\[ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{S_{\text{max}}(x)}{S_{\text{max}}(y)} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{x_i}{y_i} \]

H.W.: Follows from \( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{S_{\text{max}}(x)}{S_{\text{max}}(y)} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{x_i}{y_i} \)

Enlightened

\[ x \left( \frac{n-1}{n} \right)^{n-1} \left( \frac{n}{n} \right)^{n-1} \left( \frac{n}{n-1} \right)^{n-1} \]

\[ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{x_i}{y_i} \]

Prop: Dead r. transform

For which is zero outside \( n \rightarrow \infty \)

so for \( n = 0 \), \( S(y) \) is a density of a p.d.f. \( f(y) \)

\[ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{y_i}{S(y)} = S(x) \]

Pf: Schwarz's p.d.f. Schwarz's p.d.f. (\( L \)) is invariant to

\[ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{x_i}{y_i} \]

Markov kernel on level \( N \)

\( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{x_i}{y_i} \)

\( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{x_i}{y_i} \)
This implies that the sequential update dynamics may be constructed, and that they preserve class of Schur processes.

- Simulation - (discrete space worm process)

* If we take \( \mu = \varepsilon \times 0 \), speed up \( \varepsilon^{-1} \) we get continuous time process.  (all \( y_i = 1 \) so each level rate 1)

* \( S^\varepsilon (\varepsilon, \varepsilon, ..., \varepsilon) \rightarrow S^\varepsilon (\mu) \) Planchad specialization \( \varepsilon \varepsilon^{-1} \)

* If initially all \( x_i = 0 \), measure supported on

\[ \lambda^{(\infty)} = \cdot \cdot \cdot \lambda^{(n)} = \emptyset \]

Simulation has done affine shift to avoid overlapping

so \( (0,0,0,...) \rightarrow (0,-1,-2,-3,...) \).

See TASEP on Left edge!  (step initial cond.) push TASEP on Right edge!
Second application LPP geo. weights

\[
\begin{pmatrix}
N \\
\vdots \\
y_i \\
x_1 & x_2 & \cdots & x_N \\
\end{pmatrix} \quad w_{ij} = M \quad w_{ij} \in \mathbb{Z}_{\geq 0}
\]

Semi-discrète Poisson limits possible

equal top.

RSK bijection \( M \leftrightarrow \lambda, \mu \)

where \( \lambda^{(k)}_1 = \max_k \) \( k \)

\( \lambda^{(k)}_1 + \lambda^{(k)}_2 = \max_k \)

Ex:Prop: If \( w_{ij} \sim \text{geo}(x_i; y_j) \) then

\[
P(\lambda^{(k)}_1, \ldots, \lambda^{(k)}_N) = \frac{S_{x^{(k)}_1}(x) S_{x^{(k)}_2}(y_1) \cdots S_{x^{(k)}_N}(y_N) \cdots \mathcal{P}(x; y)}{\prod_{i<j} (x_i - x_j)}
\]

Schur process. Can increase by adding column. Give DIFFERENT Dynamic
From Schur to Macdonald

Define

\[ D_k = \frac{q}{2} \sum_{i<j} \prod_{i \in I} q^{x_i - x_j} \prod_{j \in I} q^{x_i} \]

\[(T_{x_k} f)(x_1, \ldots, x_n) = f(x_1, \ldots, q x_i, \ldots, x_n)\]

(For now set \( t = q \)). Then (from def.)

Exercise \((D_k^N s_\lambda)(x_1, \ldots, x_n) = e_k(q^\lambda_1 q^{N-1}, \ldots, q^\lambda_n q^0) s_\lambda(x_1, \ldots, x_n)\)

\[ e_k(q^\lambda_1 q^{N-1}, \ldots, q^\lambda_n q^0) = \sum_{i_1 < \ldots < i_k} x_{i_1} \ldots x_{i_k} \]

Since \( s_\lambda \) linearly span sym poly of degree \( N \), shows \( \{D_k^N\}_{k=1}^N \) are commuting operators, non-self-adjoint in inner product in which \( \langle s_\lambda, s_\mu \rangle = \delta_{\lambda=\mu} \).

Notice degeneracy (For all \( q \), \( s_\lambda \) are eigenfunctions)
They have many remarkable properties that include orthogonality with (generically) pairwise different eigenvalues

\[
\left( \begin{array}{c} \mathcal{P} \left( \sum_{N} \mathcal{P} \left( \mathcal{F} \left( \frac{1}{x-\xi} \right) \right) \right) \end{array} \right) = \left( \begin{array}{c} N \mathcal{P} \left( \mathcal{F} \left( \frac{1}{x-\xi} \right) \right) \end{array} \right) \left( \begin{array}{c} N \mathcal{P} \left( \mathcal{F} \left( \frac{1}{x-\xi} \right) \right) \end{array} \right)
\]

They diagonalize polynomials in \( N \) variables over \( \mathbb{Q}(\xi) \). They diagonalize polynomials with partitions into \( N \) variables in symmetric MacDonald polynomials with partitions into \( N \) variables in symmetric MacDonald polynomials...
At $q=t$ reduces to Schur function Cauchy identity

\[
\frac{1}{\prod_{i=1}^{\infty} (1-q^i)} = \sum_{\lambda \in \mathbb{P}} \frac{1}{\prod_{\alpha=1}^{\infty} (1-q^{\lambda_\alpha})}
\]

If $G_i \equiv \mathbb{1}$, then $\prod_{\lambda \in \mathbb{P}} \Theta_{\lambda \in \mathbb{P}} (\text{Plancherel})$.

\[
\prod_{\alpha=1}^{\infty} \left( \frac{(q; q)_\infty}{(1-q^\alpha; q)_\infty} \right) = \prod_{\alpha=1}^{\infty} \left( \frac{(q^\alpha; q)_\infty}{(1-q^{\alpha^2}; q)_\infty} \right)
\]

Reproducing kernel (Cauchy type identity)
Normal distribution

\[ \mathcal{N}(\mu, \sigma^2) \]

Macdonald polynomial

Interlacing triangular arrays (Gelfand-Tsetlin patterns)

Ascending Macdonald processes are probability measures on

\[ \text{two groups of parameters} \]

\[ \mathcal{N}(\mu, \sigma^2) \]

\[ \sum_{m=0}^{\infty} \frac{z^m}{m!} \cdot \frac{1}{g_1, g_2, \ldots, g_m} \]

\[ \frac{1}{g_1, g_2, \ldots, g_m} \cdot \frac{1}{g_1, g_2, \ldots, g_m} \]

\[ \mathcal{P}_{(N)}(z) \]

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Macdonald processes \( q, t \in [0,1) \)
- Ruijsenaars-Macdonald system
- Representations of Double Affine Hecke Algebras

q-Whittaker processes \( t=0 \)
- \( q \)-TASEP 2d dynamics
- \( q \)-deformed quantum Toda lattice
- Representations of \( \hat{q}_N, \mathcal{U}_q(\mathfrak{gl}_N) \)

General \( \beta \) RMT \( t = q^{\beta/2} \rightarrow 1 \)
- Random matrices over \( \mathbb{R}, \mathbb{C}, \mathbb{H} \)
- Calogero-Sutherland, Jack polynomials

Whittaker processes \( t = 0 \)
- Directed polymers and their hierarchies
- Quantum Toda lattice, repr. of \( GL(n, \mathbb{R}) \)

Hall-Littlewood processes \( q=0 \)
- Random matrices over finite fields
- Spherical functions for \( p \)-adic groups

Schur processes \( q=t \)
- Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE
- Characters of symmetric, unitary groups

Kingman partition structures
- Cycles of random permutations \( q=0 \)
- Poisson-Dirichlet distributions \( t=1 \)
We are able to do two basic things:

1. Evaluate averages of a rich class of observables
2. Construct explicit Markov operators that map Macdonald processes to Macdonald processes (with new parameters)

The integrable structure of Macdonald polynomials directly translates into probabilistic content.

By working at a high combinatorial level we avoid analytic issues (eventually need to work hard to take various limits).
Evaluation of averages is based on the following observation. Let $\mathcal{D}$ be an operator that is diagonalized by the Macdonald polynomials (for example, a product of Macdonald operators),

$$\mathcal{D} \, P_\lambda = d_\lambda \, P_\lambda .$$

Applying it to the Cauchy type identity $\sum_\lambda P_\lambda (a) Q_\lambda (b) = \prod (a;b)$ we obtain

$$\mathbb{E}[d_\lambda] = \frac{\mathcal{D}^{(a)} \prod (a;b)}{\prod (a;b)} .$$

If all the ingredients are explicit (as for products of Macdonald operators), we obtain meaningful probabilistic information. Contrast with the lack of explicit formulas for the Macdonald polynomials.
Expectations characterize Macdonald kernels (q,t-corr kernel):

\[ \mathcal{H}(\alpha, \omega; b_1, \ldots, b_m) = \prod_{\lambda \in \mathcal{P}_n} \prod_{\mu \in \mathcal{P}_n} \mathcal{G}(\lambda \cdot \omega; \mu) = \mathcal{E}(\alpha, \mu; \omega) \]

Macdonald difference operators

\[ \{ \mathcal{D} \} \]
Note \( \prod(a_{1}, a_{n}; b_{1}, ..., b_{m}) = \prod(a_{1}; b_{1}, ..., b_{m}) \cdots \prod(a_{n}; b_{1}, ..., b_{m}) \)

Encode products of difference operators as contour integrals

**Proposition [Borodin-C '11]:** For nice \( F(u_{1}, ..., u_{n}) = f(u_{1}) \cdots f(u_{n}) \)

\[
\left( \mathcal{D}_{N}^{\mathcal{F}} \right)(\hat{a}) = \frac{F(\hat{a})}{(2\pi i)^{r!}} \int \cdots \int \det(\frac{1}{t z_{K} - z_{E}})_{k, l = 1}^{r} \prod_{j = 1}^{n} \left( \prod_{m = 1}^{N} \frac{t z_{j} - a_{m}}{z_{j} - a_{m}} \right) \frac{f(g z_{j})}{f(z_{j})} \frac{d z_{j}}{z_{j}}
\]

Here is another example for powers of first diff. op. at \( t = 0 \)

\[
\left( \mathcal{D}_{N}^{1} \right)^{k} \mathcal{F}(\hat{a}) = (-1)^{k} \frac{q^{k-1}}{(2\pi i)^{k}} \int \cdots \int \prod_{1 \leq A < B \leq k} \frac{Z_{A} - Z_{B}}{Z_{A} - q Z_{B}} \prod_{j = 1}^{k} \left( \prod_{m = 1}^{N} \frac{a_{m}}{a_{m} - z_{j}} \right) \frac{f(g z_{j})}{f(z_{j})} \frac{d z_{j}}{z_{j}}
\]
\[
\frac{f\cdot z}{\frac{\partial}{\partial p}} \bigg[ \int_{\{(q,z): (q,z) k = 1\}} \left( \prod_{m=1}^{N - 1} \frac{1}{a_n - \frac{q}{a_m}} \right) \left( \prod_{n=1}^{\infty} \frac{1}{\gamma_{n(z)} - \frac{q}{a_n}} \right) \left( \prod_{m=1}^{N - 1} \frac{1}{a_{m,n} - \frac{q}{a_m}} \right) \right] \int_{\frac{z^2}{z^2 - \frac{q}{a_n}}}^{\infty} \frac{\Gamma(\frac{1}{k})}{\Gamma(\frac{1}{k} - \frac{q}{a_n})} \frac{\Gamma(k)}{\Gamma(k - \frac{q}{a_n})} \text{d}z
\]

\[
(\mathcal{N}_w \geq N) \Rightarrow \mathcal{N}_w \\
(\mathcal{N}_w \geq N^k) \\
(\mathcal{N}_w \geq N^k)
\]

Different levels results in the integral representation so taking products of the first order Macdonald operators (on taking t = 0, we have that

\[ q \prod_{i} P_i = q \prod_{i=1}^{N} P_i \]
At $t=0$ this reduces to the one we have already seen.

\[ \det \left( I + K \right) = \left[ \prod_{n=1}^{\infty} \left( \frac{g^n(0)}{g^n(0-1)} \right)^{b_n} \right] \left( \frac{\prod_{n=1}^{\infty} \left( \frac{g^n(0)}{g^n(0-1)} \right)^{b_n}}{\prod_{n=1}^{\infty} \left( \frac{g^n(0)}{g^n(0-1)} \right)^{b_n}} \right) \]

\((\text{Mellin-Barnes type})\)

Transform Fredholm determinant formula:

Using another operator (attributed to Noun) diagonalized by $
\begin{bmatrix} g_1 & & & \\ & \ddots & & \\ & & g_n & \\ & & & \end{bmatrix}
\]
\[ p \left( Y \right) \text{ for } Y \in \Omega \{
\begin{cases} 
0, & Y \not\in \mathcal{U}, \\
\frac{1}{n}, & Y = \mathcal{U}.
\end{cases}\right) \]

Branching rule: \[ p \left( \left( a_1, \ldots, a_n \right) \right) = \prod_{i=1}^{n} p \left( x \right) \]

[Diacomi-Filill, '09]: in Schürr process case [Borodin-Ferrari '08]

Dynamics on Gelland-Tsetlin patterns comes from the idea of
The trajectory of this Markov chain defines the Markov process

\[
W = \left( x^{(n)}_1, x^{(n)}_2, \ldots, x^{(n)}_{l-1} \right) (W_n^1, \ldots, W_n^{N}) (a_0, \ldots, a_{i-1}, a_i) [n_i] \]

maps \( W^{n-1} (a_0, \ldots, a_{i-1}, a_i) \) to \( W^n (a_0, \ldots, a_{i-1}, a_i) \).

The Markov kernel (stochastic link) from level \( N-1 \) to \( N-2 \) is
Markov kernel from level $N$ to level $N$

$$\Pi_N(u)(\mathbf{\nu}^{(N)}, \mathbf{\nu}^{(N)}) := \frac{P_{V^{(N)}}(a_1, \ldots, a_N)}{P_{\lambda^{(N)}}(a_1, \ldots, a_N)} \cdot \frac{Q_{V^{(N)}}/\lambda^{(N)}(u)}{\Pi(a_1, \ldots, a_N; u)}$$

maps $M_N(a_1, \ldots, a_N; b_1, \ldots, b_M)$ to $M_N(a_1, \ldots, a_N; b_1, \ldots, b_M, u)$.

Note: $\sum_{\mathbf{\nu}} \frac{Q_{V/\lambda}(u)}{\Pi(a_1, \ldots, a_N; u)} P_{\nu}(a_1, \ldots, a_N) = P_{\lambda}(a_1, \ldots, a_N)$

so $P_{\nu}(a_1, \ldots, a_N)$ has eigenvalue $1$ and is positive inside $\forall \lambda$ and zero outside: $(q,t)$-deformed Dyson Brownian motion
Macdonald processes [Borodin–Petrov, 13]

Other dynamics also preserve class of

\[ \text{mapping} \]

sequentially updating GT-pattern, mapping

\[ \prod_{(i-k) \in N} (\lambda_i, \lambda_j) \]

\[ \begin{align*}
\prod_{(i-k) \in N} (\lambda_i, \lambda_j) \\
\end{align*} \]

\[ \prod_{(i-k) \in N} (\lambda_i, \lambda_j) \]

Multivariate Markov kernel
where

\[ b_w X^{w+(+)} b_w = X b_w \]

forms $q$-TASEP

\[ \text{The set of coordinates} \left\{ (w-1)_w, \ldots, m_w \right\} \]

\[ \text{Simulation} \]

\[ \text{Each coordinate jumps by } 1 \text{ to the right independently of the others with rate } (1-w)_w. \]

\[ \text{Whittaker processes (Macdonald process with } t=0). \]

Here is an example of a Markov process preserving the class of the
After time $\tau$, Plancherel specialization with $\gamma = \tau$, hence

$$\prod(a_1, \ldots, a_N; \text{Plan}_\tau) = \prod_{i=1}^{N} e^{\tau a_i}$$

$$a_i \equiv 1$$

$x_3(t)$ $x_2(t)$ $\text{gap} = 3$ $x_4(t)$

rate $1 - q^{\text{gap}}$

$$q^{\lambda(m)} = q^{x_m(t) + m}$$

$0 < q < 1$

**Theorem [Borodin-C '11, [B-C-Sasamoto '12]**  For $q$-TASEP with $\{X_n(0) = -n\}_{n \geq 1}$

$$E\left[q^{\left(\sum_{k=1}^{N_k} (x_{N_1} + N_1) + \ldots + (x_{N_k} + N_k)\right)}\right] = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \oint \cdots \oint \prod_{A < B} \frac{z_A z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)\tau z_j}}{(1 - z_j)^{N_j}} \frac{dz_j}{z_j}$$

$(N_1 \geq N_2 \geq \ldots \geq N_k)$

$\ast 0 \left(1 \cdots z_{k-1} \begin{array}{c} \text{cycle} \\ z_k \end{array} \right) z_1$
The real array, low rows of the triangular array behave as

\[ \text{as } \theta = \frac{\pi}{2}, \text{ at large times } t/\theta, \text{ with zero initial conditions,} \]
Partition function for a semi-discrete directed random polymer

\[ z(t,N) = \int e^{B_1(0,s_1) + B_2(s_1,s_2) + \cdots + B_N(s_{N-1},t)} ds_1 \cdots ds_{N-1} \]

0 < s_1 < \cdots < s_{N-1} < t

\( B_1, \ldots, B_N \) independent Brownian motions

\[ B_k(\alpha, \beta) := B_k(\beta) - B_k(\alpha) = \int_{\alpha}^{\beta} B_k(x) \, dx \]

Whittaker measure arises under geometric lifting of RSK (tropical)

Also: log-gamma polymer [C-O'Connell-Seppalainen-Zygouras '11]

Different dynamics than Diaconis-Fill! Fixed level -> quantum Toda

Initially unclear how to take asymptotics of this.

\[
\left( \frac{1}{2\pi N} \right)^{\frac{1}{4}} \prod_{i=1}^{N} \frac{1}{\prod_{j \neq i}^{N} \sqrt{1 - \theta^2 j^2}} = N! \left( \frac{\pi}{2} \right)^N
\]

O'Connell, 09] proved a Laplace transform formula
To summarize:

- ASEP and q-TASEP are important systems in the KPZ universality class, which can be scaled to the KPZ equation
- Macdonald processes are a source of integrable probabilistic models
- Generalize Schur processes but are not determinantal
- Integrability from structural properties of Macdonald polynomials (lead to nice Markov dynamics and concise formulas for averages)
- Turning averages into asymptotics remains challenging
- Rigorous replica trick developed for q-TASEP and ASEP
- Nested contour integral ansatz formulas for ASEP moments suggest search for new structure parallel to Macdonald processes
universal limits ( Tracy-Widom distributions, Airy processes)

KPZ/SHE/continuous Brownian polymer

polymers

semi-discrete Brownian polymer

Log-Cauchy discrete polymer

ASEP

q-ASEP

q-pushASEP

Discrete time q-TASEPs
Macdonald processes \( q, t \in [0, 1) \)
Ruijsenaars-Macdonald system
Representations of Double Affine Hecke Algebras

q-Whittaker processes \( t=0 \)
- q-TASEP, 2d dynamics
- q-deformed quantum Toda lattice
- Representations of \( \hat{q}_N \), \( U_q(\mathfrak{gl}_n) \)

Hall-Littlewood processes \( q=0 \)
Random matrices over finite fields
Spherical functions for p-adic groups

General \( \beta \) RMT \( t=\frac{\beta t}{2} \rightarrow 1 \)
Random matrices over \( \mathbb{R}, \mathbb{C}, \mathbb{H} \)
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Whittaker processes \( t \rightarrow 1 \)
- Directed polymers and their hierarchies
- Quantum Toda lattice, repr. of \( GL(n, \mathbb{R}) \)

Kingman partition structures
Cycles of random permutations \( q=0 \)
Poisson-Dirichlet distributions \( t=1 \)

Schur processes \( q=t \)
Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE
Characters of symmetric, unitary groups
A few directions:

- General initial data (diagonalize many-body systems)?
- Symmetries (half-space polymers, Koornwinder processes)?
- Multipoint/multitime asymptotics?
- Other solvable systems (e.g. discrete time $q$-PushASEP/ASEP)?
- RSK type correspondences at $(q,t)$ level?
- ASEP 2+1 extension (analog of Macdonald processes)?
- Higher versions of Macdonald processes?
- Other degenerations?
- KPZ fixed point / equation universality?
Macdonald processes (arXiv:1311.4408)
(arXiv:1207.5035)
From duality to determinants for g-TASEP and ASEPs
Articles:
Two ways to solve ASEPs (arXiv:1212.2267)
(arXiv:1106.1596)
The Kardar-Parisi-Zhang equation and universality class
Lectures on integrable probability (arXiv:1212.3351)
Reviews: