Due in class Thursday, Jan. 28.

We encourage collaboration on homework in this course. However, if you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through translate to poor grades on exams. You must turn in your own writeups of all problems, and, if you do collaborate, you must write on the front of your solution sheet the names of the students you worked with.

Because the solutions will be available immediately after the problem sets are due, no extensions will be possible.

Each problem set is graded out of 80 points.

4. (a) (i) Use the $s$-shift rule to provide rules computing the Laplace transform of $\cos(\omega t) f(t)$ and $\sin(\omega t) f(t)$ in terms of the Laplace transform of $f(t)$. Use this to compute $\mathcal{L}(t^2 \cos(\omega t))$, and compare your result with what you get from the $t^n$-rule $t^n f(t) \leadsto (-1)^n F^{(n)}(s)$.

Compute the Laplace transform of each of the following functions.

(ii) $t^2 + 1$
(iii) $(t^2 + 1)\sin(2t)$
(iv) $(t^2 + 1)e^{-t}\sin(2t)$.

(b) For this problem we will consider two systems:

System 1: $(D^2 + 2D + 4I)x = (2D + 4I)f$

System 2: $(D^2 + 8D + 7I)x = f$.

In both systems $f(t)$ is considered the input and $x(t)$ the response.

Suppose we make a cascading system by taking the output from System 1 and using it as the input to system 2.

(i) Draw a block diagram for the cascade.
(ii) What is the transfer function for the cascading system?
(iii) What is the complex gain of the cascade, for sinusoidal input of angular frequency $\omega$? (A function of $\omega$.)
(iv) What is the (real) gain? (Again, a function of $\omega$.)
(v) Draw the pole/zero diagram for the system.

(c) Invoke the Mathlet Bode and Nyquist Plots
http://mathlets.org/mathlets/bode-and-nyquist-plots/

This Mathlet allows you to construct a rational function $G(s)$ by specifying its zero/pole diagram. (The numerator and denominator of $G(s)$ are normalized to be 1.) The Bode plots (either linear or log scale) of the corresponding system – that is, the graphs of the gain $|G(i\omega)|$ and the phase lag $-\text{Arg}(G(i\omega))$ can be involved using radio buttons at the bottom. The horizontal scale of the pole diagram can be adjusted using the slider below it.
All radio spam has been restricted to frequencies in the range $[2, 3]$ (in units of megahertz, say). Make a sketch of an ideal filter that has the effect of completely suppressing those frequencies but allowing all other frequencies to pass with gain 1. Use your understanding of how to relate the pole/zero diagram to the gain curve to design a filter with 4 poles and 4 zeros that will suppress spam signals, but allow other frequencies to pass.

Why is it advantageous to have the same number of zeros and poles? What effect does this have on the gain curve?

Discuss the design issues you encounter: what trade-offs did you face? You may want to suggest several alternative designs, optimizing in different ways.

Print a screen shot of your result(s) and turn it in with this pset.

5. (a) More practice with Laplace Transform: Find the Laplace transform of each of the following functions.
   (i) $f(t) = u(t - 1)t$.
   (ii) $f(t) = \begin{cases} t & \text{for } 0 < t < 1 \\ 1 - t & \text{for } t > 1 \end{cases}$

   (b) Let $f(t)$ be the square-wave of period $2P$:
   $$f(t) = \begin{cases} 1 & \text{for } 0 < t < P \\ -1 & \text{for } P < t < 2P \end{cases}$$
   and $f(t + 2P) = f(t)$.
   (i) Sketch a graph of $f(t)$, and express $f(t)$ (for $t > 0$) as a linear combination of shifted step functions.
   (ii) Sketch a graph of $f'(t)$, and express it as a linear combination of shifted $\delta$ functions.
   (Assume that $f(0-) = -1$.)
   (iii) Compute the Laplace transform of $f'(t)$. Rewrite your answer using the geometric series. Then use the $t$ derivative rule to compute the Laplace tranform of $f(t)$.

   (c) Find the inverse transform of the following functions in the frequency domain.
   (i) $\frac{s^2}{s^2 + 1}$
   (ii) $\frac{e^{-s}}{s^3 + s}$.

   (d) Suppose that $F(s) = \mathcal{L}(f(t); s)$. Let $a > 0$ and let $g(t) = f(at)$. Find $\mathcal{L}(g(t); s)$ in terms of $F(s)$. Check your formula using the example $f(t) = t^a$.

   (e) Compute the impulse and step responses of the system you analyzed on the first day of class – the one illustrated by the Mathlet
   \url{http://mathlets.org/mathlets/amplitude-and-phase-2nd-order-ii/}
   Amplitude and Phase Second Order II – with the values $m = 2$, $b = 2$, $k = 10$. What is the value of the step response for large $t$? Does that value make sense?

6. (a) Consider the spring/mass/dashpot system, driven through the spring (as in the Mathlet \url{http://mathlets.org/mathlets/amplitude-and-phase-2nd-order/} Amplitude
and Phase Second Order I); the differential equation is $m\ddot{x} + b\dot{x} + kx = ky$ where $x$ is the system response and $y$ is the input signal. Draw closed loop block diagrams representing each of the following perspectives on this system as exhibiting feedback.

(i) The open loop system has unstable system function $\frac{k}{ms^2 + bs}$.

(ii) The open loop system just lacks the damping term; the feedback (control) is provided by the damping term. For this, it may be useful to regard the open loop system as given by a cascade of $k$ followed by $\frac{1}{ms^2 + k}$.

Check your work by applying Black’s formula in each case.

(b) A certain amplifier has system function $\frac{ca}{s + a}$, where $c$ and $a$ are positive real constants.

(i) Is this system stable? How many independent transients are there for this system? What are they? Express them in terms of exponentials written as $e^{-t/t_0}$ (perhaps times sinusoids); $t_0$ is the “characteristic time” of this decaying exponential.

(ii) Reverse engineer this to get a differential equation relating the input signal $y$ to the system response $x$.

(iii) Compute the gain function of this system. At what frequency $\omega_r$ is the gain maximal? What is the gain at $\omega = \omega_r$? Estimate the gain for large $\omega$ (as a function of $\omega$).

(iv) This system may be regarded as a low-pass filter. The “half-power point” is the frequency $\omega_{1/2}$ at which the gain is $\frac{1}{\sqrt{2}}$ times the maximal gain (because the power scales as the square of the amplitude). What is the half power frequency of this system?

(v) Compute the unit step response and the unit impulse response.

(vi) Now add a negative feedback branch with proportional feedback; that is, the feedback system function is a constant $k$. Carry out steps (i)–(v) for this new system. Please use the notation $b = a(1 + ck)$ whenever it’s convenient.

Lesson: Adding the feedback loop decreases the maximal gain, but broadens the pass-band and decreases the characteristic time.

7. (a) Compute $f(t) * u(t)$ (where $u(t)$ denotes the Heaviside step function). Explain your result by regarding $f(t)$ as the input signal for a system with weight function $u(t)$. By the way, what is the differential operator with weight function $u(t)$?

(b) Laplace transform can be used to verify many properties of convolution; for example, it is associative and commutative, and $\delta(t)$ serves as a unit (so $\delta * f = f$). Use the Laplace transform to verify the surprising fact that if $f(0-) = g(0-) = 0$ then

$$f' * g = f * g' = (f * g)'$$

Check the first of these equalities by integration. What’s the formula for $f' * g - f * g'$ when $f(0-)$ and $g(0-)$ are not necessarily zero?

(c) Use the Laplace transform to compute $\frac{t^m}{m!} * \frac{t^n}{n!}$.

(d) Use the fact that the RIC response to an LTI system with weight function $w(t)$ to an input signal $f(t)$ is given by $f(t) * w(t)$ to sketch $f(t) * w(t)$ for each of the following pairs:
(i) \( f(t) = 2u(t) \), \( w(t) = e^{-t} \).

(ii) \( f(t) \) the “square wave” described in 5(b) above, \( w(t) = e^{-t} \). For \( f(t) \) let \( P = \pi \).

(iii) \( f(t) = \sin(t) \), \( w(t) = \sin(t) \).