1 Introductory example

Consider the system \( \dot{x} + 3x = f(t) \), where \( f \) is the input and \( x \) the response. We know its unit impulse response is

\[
\begin{cases} 
0 & \text{for } t < 0 \\
e^{-3t} & \text{for } t > 0
\end{cases} = u(t)e^{-3t}.
\]

This is the response from rest IC to the input \( f(t) = \delta(t) \). What if we shifted the impulse to another time, say, \( f(t) = \delta(t - 5) \)? Linear time invariance tells us the response will also be shifted. That is, the solution to

\[
\dot{x} + 3x = \delta(t - 2), \quad \text{with rest IC}
\]

is

\[
x(t) = w(t - 2) = \begin{cases} 
0 & \text{for } t < 2 \\
e^{-3t} & \text{for } t > 2
\end{cases} = u(t - 2)e^{-3(t-2)}.
\]

In words, this is a system of exponential decay. The decay starts as soon as there is an input into the system. Graphs are shown in Figure 1 below.

![Graphs of \( w(t) \) and \( x(t) = w(t - 2) \).](image)

We know that \( \mathcal{L}(\delta(t-a)) = e^{-as} \). So, we can find \( X = \mathcal{L}(x) \) by taking the Laplace transform of Equation 1.

\[
(s + 3)X(s) = e^{-2s} \Rightarrow X(s) = \frac{e^{-2s}}{s + 3} = e^{-2s}W(s),
\]

where \( W = \mathcal{L}w \). So delaying the impulse until \( t = 2 \) has the effect in the frequency domain of multiplying the response by \( e^{-2s} \). This is an example of the \( t \)-translation rule.

2 \( t \)-translation rule

The \( t \)-translation rule, also called the \( t \)-shift rule gives the Laplace transform of a function shifted in time in terms of the given function. We give the rule in two forms.

\[
\mathcal{L}(u(t - a)f(t-a); s) = e^{-as}F(s) \quad (2)
\]

\[
\mathcal{L}(u(t - a)f(t); s) = e^{-as}\mathcal{L}(f(t + a); s). \quad (3)
\]

For completeness we include the translation formulas for \( u(t - a) \) and \( \delta(t-a) \):

\[
\mathcal{L}(u(t - a)) = e^{-as}/s \quad (4)
\]

\[
\mathcal{L}(\delta(t-a)) = e^{-as}. \quad (5)
\]
Remarks:
1. Formula 3 is ungainly. The notation will become clearer in the examples below.
2. Formula 2 is most often used for computing the inverse Laplace transform, i.e., as
   \[ u(t - a)f(t - a) = \mathcal{L}^{-1}(e^{-as}F(s)). \]
3. These formulas parallel the s-shift rule. In that rule, multiplying by an exponential on
   the time (t) side led to a shift on the frequency (s) side. Here, a shift on the time side leads
   to multiplication by an exponential on the frequency side.

Proof: The proof of Formula 2 is a very simple change of variables on the Laplace integral.

\[
\mathcal{L}(u(t - a)f(t - a); s) = \int_{0}^{\infty} u(t - a)f(t - a)e^{-st} dt \\
= \int_{a}^{\infty} f(t - a)e^{-st} dt \quad (u(t - a) = 0 \text{ for } t < a) \\
= \int_{0}^{\infty} f(\tau)e^{-s(\tau + a)} d\tau \quad \text{(change of variables: } \tau = t - a) \\
= e^{-as} \int_{0}^{\infty} f(\tau)e^{-s\tau} d\tau \\
= e^{-as}F(s).
\]

Formula 3 follows easily from Formula 2. The easiest way to proceed is by introducing a
new function. Let \( g(t) = f(t + a) \), so

\[ f(t) = g(t - a) \quad \text{and} \quad G(s) = \mathcal{L}(g) = \mathcal{L}(f(t + a)). \]

We get

\[ \mathcal{L}(u(t - a)f(t); s) = \mathcal{L}(u(t - a)g(t - a)) = e^{-as}G(s) = e^{-as}\mathcal{L}(f(t + a); s). \]

The second equality follows by applying Formula 2 to \( g(t) \).

Example 1. Find \( \mathcal{L}^{-1}\left(\frac{\omega e^{-as}}{s^2 + \omega^2}\right) \).

answer: First ignore the exponential and let

\[ f(t) = \mathcal{L}^{-1}\left(\frac{\omega}{s^2 + \omega^2}\right) = \sin(\omega t). \]

Using the shift Formula 2 this becomes

\[ \mathcal{L}^{-1}\left(\frac{\omega e^{-as}}{s^2 + \omega^2}\right) = u(t - a)f(t - a) = u(t - a)\sin(\omega(t - a)). \]

Example 2. \( \mathcal{L}(u(t - 3)t; s) = e^{-3s}\mathcal{L}(t + 3; s) = e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right). \)

Example 3. \( \mathcal{L}(u(t - 3) \cdot 1; s) = e^{-3s}\mathcal{L}(1; s) = e^{-3s}/s. \)
Example 4. Find \( L(f) \) for \( f(t) = \begin{cases} 0 & \text{for } t < 2 \\ t^2 & \text{for } t > 2 \end{cases} \).

answer: In order to use the \( t \)-shift rule we have to write \( f(t) \) in \( u \)-format:

\[ f(t) = u(t - 2)t^2. \]

So, Formula 3 says

\[ L(f) = e^{-2s}L((t + 2)^2; s) = e^{-2s} \left( \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \right). \]

Example 5. Find \( L(f) \) for \( f(t) = \begin{cases} \cos(t) & \text{for } 0 < t < 2\pi \\ 0 & \text{for } t > 2\pi \end{cases} \).

answer: Again we first put \( f(t) \) in \( u \)-format. Notice the the function

\[ u(t) - u(t - 2) = \begin{cases} 1 & \text{for } 0 < t < 2 \\ 0 & \text{elsewhere}. \end{cases} \]

Therefore

\[ f(t) = (u(t) - u(t - 2\pi)) \cos(t) = u(t) \cos(t) - u(t - 2\pi) \cos(t). \]

Using the \( t \)-translation formula 3 we get

\[ L(f) = \frac{s}{s^2 + 1} - e^{-2\pi s}L(\cos(t + 2\pi)) = \frac{s}{s^2 + 1} - e^{-2\pi s} \frac{s}{s^2 + 1}. \]

The last equality holds because \( \cos(t + 2\pi) = \cos(t) \).

3 A longer example

The fish population in a lake is not reproducing fast enough and the population is decaying exponentially with decay rate \( k \). A program is started to stock the lake with fish. Three different scenarios are discussed below.

Example 6. A program is started to stock the lake with fish at a constant rate of \( r \) units of fish/year. Unfortunately, after 1/2 year the funding is cut and the program ends. Model this situation and solve the resulting DE for the fish population as a function of time.

answer: Let \( x(t) \) be the fish population and let \( A = x(0^-) \) be the initial population. Exponential decay means the population is modeled by

\[ \dot{x} + kx = f(t), \quad x(0^-) = A \quad (6) \]

where \( f(t) \) is the rate fish are being added to the lake. In this case

\[ f(t) = \begin{cases} r & \text{for } 0 < t < 1/2 \\ 0 & \text{for } 1/2 < t. \end{cases} \]
First we write \( f \) in \( u \)-format: 
\[ f(t) = r(1 - u(t - 1/2)) \]
and find the Laplace transform of the equation.
\[ F(s) = \mathcal{L}(f)(s) = \frac{r}{s} - \frac{r}{s} e^{-s/2}. \]

Next we find the Laplace transform of the equation and solve for \( f \).
\[
\begin{align*}
    sX - x(0^-) + kX &= F(s) \\
    (s + k)X - A &= \frac{r}{s}(1 - e^{-s/2}) \\
    X(s) &= \frac{A}{s + k} + \frac{r}{s(s + k)}(1 - e^{-s/2}).
\end{align*}
\]

To find \( x(t) \) we temporarily ignore the factor of \( e^{-s/2} \) and take Laplace inverse of what’s left. (using partial fractions).
\[
\begin{align*}
    \mathcal{L}^{-1}\left(\frac{A}{s + k}\right) &= Ae^{-kt}, \\
    \mathcal{L}^{-1}\left(\frac{r}{s(s + k)}\right) &= \frac{r}{k}(1 - e^{-kt}).
\end{align*}
\]

The \( t \)-translation formula then says
\[
\mathcal{L}^{-1}\left(\frac{re^{-s/2}}{s(s + k)}\right) = \frac{r}{k}u(t - 1/2)\left(1 - e^{-k(t-1/2)}\right).
\]

Putting it all together we get (in \( u \) and cases format).
\[
x(t) = Ae^{-kt} + \frac{r}{k}\left(1 - e^{-kt}\right) - \frac{r}{k}u(t - 1/2)\left(1 - e^{-k(t-1/2)}\right)
\]
\[
= \begin{cases} 
    Ae^{-kt} + \frac{r}{k}\left(1 - e^{-kt}\right) & \text{for } 0 < t < 1/2 \\
    Ae^{-kt} - \frac{r}{k}\left(e^{-kt} + e^{-k(t-1/2)}\right) & \text{for } 1/2 < t.
\end{cases}
\]

**Example 7.** (Periodic on/off) The program is refunded and they have enough money to stock at a constant rate of \( r \) for the first half of each year. Find \( x(t) \) in this case.

**answer:** All that’s changed from Example 6 is the input function \( f(t) \). We write it in cases-format and translate that to \( u \)-format so we can take the Laplace transform.

\[
\begin{align*}
    f(t) &= \begin{cases} 
        r & \text{for } 0 < t < 1/2 \\
        0 & \text{for } 1/2 < t < 1 \\
        r & \text{for } 0 < t < 3/2 \\
        0 & \text{for } 3/2 < t < 2 \\
        \ldots
    \end{cases} \\
    &= r\left(1 - u(t - \frac{1}{2}) + u(t - 1) - u(t - \frac{3}{2}) + \ldots\right)
\end{align*}
\]

The computations from here are essentially the same as in the previous example. We sketch them out.
\[
\mathcal{L}(f) = \frac{r}{s}\left(1 - e^{-s/2} + e^{-s} - e^{-3s/2} + \ldots\right), \quad \text{so } \quad X = \frac{A}{s + k} + \frac{r}{s(s + k)}\left(1 - e^{-s/2} + e^{-s} - \ldots\right).
\]
Thus,

\[ x(t) = Ae^{-kt} + \frac{r}{k} \left[ (1 - e^{-kt}) - u(t - 1/2)(1 - e^{-k(t-1/2)}) + u(t - 1)(1 - e^{-k(t-1)}) - u(t - 3/2)(1 - e^{-k(t-3/2)}) + \ldots \right] \]

and in cases format:

\[
x(t) = \begin{cases} 
Ae^{-kt} + \frac{r}{k} - \frac{r}{k} e^{-kt} & \text{for } 0 < t < \frac{1}{2} \\
Ae^{-kt} - \frac{r}{k} (e^{-kt} - e^{-k(t-1/2)}) & \text{for } \frac{1}{2} < t < 1 \\
\ldots \\
Ae^{-kt} + \frac{r}{k} - \frac{r}{k} (e^{-kt} - e^{-k(t-1/2)} + \ldots + e^{-k(t-n/2)}) & \text{for } n < t < n + \frac{1}{2} \\
Ae^{-kt} - \frac{r}{k} (e^{-kt} - e^{-k(t-1/2)} + \ldots - e^{-k(t-n-1/2)}) & \text{for } n + \frac{1}{2} < t < n + 1 \\
\ldots 
\end{cases}
\]

Note that the pattern in the formula for the response alternates between the periods of stocking and not stocking. In particular, notice that the constant term \(r/k\) is only present during periods of stocking.

**Example 8.** (Impulse train) The answer to the previous example is a little hard to read. We know from experience that impulsive input usually leads to simpler output. In this scenario suppose that once a year \(r/2\) units of fish are dumped all at once into the lake. Find \(x(t)\) in this case.

**Answer:** Once again, all that’s changed from Example 6 is the input function \(f(t)\). In this case we have

\[ f(t) = \frac{r}{2} \left( \delta(t) + \delta(t - 1) + \delta(t - 2) + \delta(t - 3) + \ldots \right). \]

This is called an impulse train. Its Laplace transform is easy to find.

\[ F(s) = \frac{r}{2} \left( 1 + e^{-s} + e^{-2s} + e^{-3s} + \ldots \right). \]

One nice thing about delta functions is that they don’t introduce any new terms into the partial fractions part of the problem.

\[ sX(s) - x(0^-) + kX(s) = \frac{r}{2} \left( 1 + e^{-s} + e^{-2s} + e^{-3s} + \ldots \right). \]

\[ \Rightarrow X(s) = \frac{A}{s + k} + \frac{r}{2(s + k)} \left( 1 + e^{-s} + e^{-2s} + e^{-3s} + \ldots \right). \]

Laplace inverse is easy:

\[ L^{-1} \left( \frac{1}{s + k} \right) = e^{-kt} \quad \Rightarrow \quad L^{-1} \left( \frac{e^{-ns}}{s + k} \right) = u(t - n)e^{-k(t-n)}. \]

Thus,

\[ x(t) = Ae^{-kt} + \frac{r}{2} e^{-kt} + \frac{r}{2} u(t - 1)e^{-k(t-1)} + \frac{r}{2} u(t - 2)e^{-k(t-2)} + \frac{r}{2} u(t - 3)e^{-k(t-3)} + \ldots \]

Here are graphs of the solutions to examples 6 and 8 (with \(A = 0, k = 1, r = 2\)). Notice how they settle down to periodic behavior.
Fig. 1. Graphs from example 2 (left) and example 3 (right).