18.024 Practice Problems for Final

1. Find the volume of the region bounded by \( z = x^2 + y^2 \) and \( z = 5 - x^2 - 2y^2 \).
2. Find the volume of the region described by \( x^2 + y^2 \leq a^2 \) and \( x^2 + z^2 \leq a^2 \).
3. Compute the triple integral \( \int_0^1 \int_0^x f(x, y, z) \, dz \, dy \) where \( W \) is the pyramid with top vertex \((0,0,1)\) and the base vertices at \((0,0,0), (1,0,0), (1,1,0)\).
4. Consider the integral
   \[
   \int_0^1 \int_0^x \int_0^y f(x, y, z) \, dz \, dy.
   \]
   Write the integral with the integration order \( dx \, dy \, dz \).
5. Use polar coordinates to evaluate \( \int_R \sqrt{x^2 + y^2} \, dx \, dy \) where \( R = [0, 1] \times [0, 1] \).
6. Compute \( \int_R (x+y)^2 e^{x+y} \, dx \, dy \) where \( R \) is the region bounded by \( x+y=1, x+y=4, \) \( x-y=-1 \) and \( x-y=1 \).
7. Consider the parametrized surface \( \Phi(r, \theta) = (r \cos \theta, r \sin \theta, \theta) \), \( 0 \leq r \leq 1 \), \( 0 \leq \theta \leq 4\pi \).
   Sketch and describe the surface. Find an expression for a “unit” normal to the surface and find the area of the surface.
8. Show that the surface \( z = (x^2 + y^2)^{-1/2} \), where \( 1 \leq z < \infty \) can be filled but not painted! (i.e. the volume under the graph is finite but the surface area is infinite!)
9. Let \( S \) be the sphere of radius \( R \). Compute the integral \( \int_S (x^2 + y^2 + z^2) \, dS \) (This should be “very” easy). Use a clever idea to evaluate \( \int_S x^2 \, dS \).
10. Let \( S \) be the sphere of radius \( R \) and \( P \) be a point inside the sphere but not on it. Show that \( \int_S \frac{1}{||X-P||} \, dS = 4\pi R \). Here \( X = (x,y,z) \).
11. Let \( S \) be part of the cone \( z^2 = x^2 + y^2 \) with \( z \) between 1 and 2 oriented such that the unit normal goes out of the cone. Compute \( \int_S F \cdot n \, dS \) where \( F(x,y,z) = (x^2, y^2, z^2) \).
12. Find \( \int_S (\nabla \times F) \cdot n \, dS \) where \( S \) is the ellipsoid \( x^2 + y^2 + 5z^2 = 16 \) and \( F = (\sin(xy), e^{x+y}, xy^3) \). (Here the unit normal goes out of \( S \))
13. Let \( F = (y,-x,zx^3y^2) \). Evaluate \( \int_S (\nabla \times F) \cdot n \, dS \) where \( S \) is the surface \( x^2+y^2+z^2 = 1, z \leq 0 \) with the unit normal going out of the sphere.
14. Let \( F = (x^3,y^3,z^3) \). Compute the flux of \( F \) going out the unit sphere.
15. Let \( S \) be a closed surface which is the surface boundary of a solid \( V \) in \( \mathbb{R}^3 \). Evaluate \( \int_S (\mathbf{r} \cdot n) \, dS \) where the normal is outward.
16. Evaluate the surface integral \( \int_S F \cdot n \, dS \) where \( F = (1,1,z(x^2+y^2)^2) \) and \( S \) is the surface of the cylinder \( x^2+y^2 \leq 1, 0 \leq z \leq 1 \). (Here the unit normal goes inside the cylinder.)
17. Prove Gauss’ Law: Let $V$ be a bounded solid which is bounded by a smooth orientable closed surface $S = \partial V$. Assume $(0, 0, 0)$ is not on $S$. Then the flux
\[ \int \int_S \vec{r} \cdot \vec{n} \frac{dS}{r^3} \]
is zero if $(0, 0, 0) \notin V$ and it is equal to $4\pi$ if $(0, 0, 0) \in V$. 