In these notes we prove the theorem stated without proof in Ch. IV, Exercises and Further Results, C6, p 489. The theorem describes the precise image of the Schwartz space \( \mathcal{S}(G) \) under the spherical transform. The proof is a modification of the proof by Anker [1991], a proof which included also the generalizations to \( \mathcal{S}^p(G) (0 < p \leq 2) \) described on p. 489. Anker’s proof was much simpler than the preceding ones and was accomplished by a skillful use and extension of the Paley–Wiener theorem (Ch. IV, Theorem 7.1) for the spherical transform. For the case \( p = 2 \) we shall here simplify the proof a bit further.

We shall use notation from the text (mainly Ch. IV) without repetition of definition.

## 1 Spherical Functions

Here we prove some estimates from Harish–Chandra [1958a] of the spherical function and its derivatives.

**Theorem 1.1.** Let \( \varphi_\lambda \) denote the spherical function

\[
\varphi_\lambda(g) = \int_K e^{i(\lambda - \rho)(H(gk))} \, dk \quad \lambda = \text{Re } \lambda + i \text{ Im } \lambda.
\]

Then we have the following estimates:

(i) \( e^{-\rho(H)} \leq \varphi_0(\exp H) \leq c(|H| + 1)^d e^{-\rho(H)} \), \( H \in \mathfrak{a}^+ \), where \( c \) is a constant, \( d = \text{Card } (\Sigma^+_0) \).

(ii) \( 0 \leq \varphi_{-i\lambda}(H) \leq e^{\lambda(H)} \varphi_0(\exp H), \quad H \in \mathfrak{a}^+ \), \( \lambda \in \mathfrak{a}^*_+ \).

(iii) Given \( D \in \mathfrak{d}(G) \) there is a constant \( c > 0 \) such that

\[
|\langle D\varphi_\lambda \rangle(g)| \leq c(|\lambda| + 1)^{\deg D} \varphi_{i\text{Im } \lambda}(g).
\]

(iv) Given a polynomial \( P \in S(\mathfrak{a}^*) \) there is a constant \( c > 0 \) such that

\[
\left| P \left( \frac{\partial}{\partial \lambda} \right) \varphi_\lambda(g) \right| \leq c(|g| + 1)^{\deg P} \varphi_{i\text{Im } \lambda}(g),
\]

where \( |g| = |H| \) if \( g = k_1 \exp H k_2 \), \( H \in \mathfrak{a} \).

**Proof:**

(i) This part is Exercise IV, B1, and the proof is on p. 580. See also Harish–Chandra [1958a], p. 279 for the original proof.
(ii) We have

\[ (1.2) \quad \varphi_{-i\lambda}(a) = \int_{K} e^{(\lambda - \rho)(H(ak))} ak \leq e^{\lambda(\log a)} \varphi_0(a) \quad a \in A^+ \]

by IV, Lemma 6.5.

(iii) According to Ch. IV, Lemma 4.4,

\[ (1.3) \quad \varphi_\lambda(gh) = \int_{K} e^{(-i\lambda + \rho)(A(kg^{-1}))} e^{(i\lambda + \rho)(A(kh))} \, dk. \]

Let \( X \in \mathfrak{g} \) and \( \bar{X} \) the corresponding left-invariant vector field. Put \( h = \exp tX \) in (1.3) and take \((d/dt)_0\). We have \( A(k \exp tX) = \exp tAd(k)X \) so

\[ (\bar{X}\varphi_\lambda)(g) = \int_{K} e^{(-i\lambda + \rho)(A(kg^{-1}))} ((Ad(k)\bar{X})\eta_\lambda)(e) \, dk, \]

where \( \eta_\lambda(h) = e^{(i\lambda + \rho)(A(h))} \). More generally, if \( D \in \mathbf{D}(G) \),

\[ (D\varphi_\lambda)(g) = \int_{K} e^{(-i\lambda + \rho)(A(kg^{-1}))} ((Ad(k)D)\eta_\lambda)(e) \, dk. \]

Since \( \eta_k \) is left \( N \)-invariant and right \( K \)-invariant we have by Ch. II, Lemma 5.14,

\[ (1.5) \quad (Ad(k)D\eta_\lambda)(e) = ((Ad(k)D)_a\bar{\eta}_\lambda)(e), \]

the bar denoting restriction to \( A \).

If \( \ell = \deg D \) we fix a basis \( D_1, \ldots, D_m \) of \( \mathbf{D}_\ell(A) \), the space of elements in \( \mathbf{D}(A) \) of degree \( \leq \ell \).

Then

\[ (Ad(k)D) = \sum_{i=1}^{m} \eta_i(k)D_i, \quad \eta_i \in \mathcal{E}(K). \]

Thus expression (1.5) reduces to

\[ \sum_{i=1}^{m} \eta_i(k)D_i(i\lambda + \rho) \]

so the right hand side of (1.4) is majorized by

\[ c \int_{K} e^{(\Im \lambda + \rho)(A(kg^{-1}))} \, dk \quad (|\lambda| + 1)\ell \quad (c = \text{const.}). \]

Since (by (1.3)) \( \varphi_\mu(g^{-1}) = \varphi_{-\mu}(g) \), this proves (iii). Harish–Chandra’s original proof is in his paper [1958a], p. 294.

For (iv) we observe from (1.1) that

\[ P \left( \frac{\partial}{\partial \lambda} \right) \varphi_\lambda(g) = \int_{K} e^{(i\lambda - \rho)(H(gk))} P(iH(gk)) \, dk \]

and now the result follows from IV §10, (14).
2 The Schwartz Spaces

The Schwartz space $I^2(G)$ consists of the $K$-bi-invariant functions $f \in E(G)$ for which the seminorm

$$\sigma_{D,q}(f) = \sup_g (|g| + 1)^q \phi_0(g)^{-1}|Df(g)|$$

is finite for each $q \in \mathbb{Z}^+$, $D \in \mathcal{D}(G)$. With these seminorms, $I^2(G)$ is a Fréchet space.

We consider the following transforms: The spherical transform $\mathcal{F} : f \rightarrow \tilde{f}$ given by

$$\tilde{f}(\lambda) = \int_G f(g) \varphi_\lambda(g) \, dg, \quad f \text{ $K$-bi-invariant},$$

the Euclidean Fourier transform $\mathcal{F}_0 : \varphi \rightarrow \varphi^*$ given by

$$\varphi^*(\lambda) = \int_A \varphi(a)e^{-i\lambda (\log a)} \, da,$$

and the Abel transform $f \rightarrow Af$ given by

$$(Af)(a) = e^{\rho(\log a)} \int_N f(an) \, dn, \quad f \text{ $K$-bi-invariant}.$$

We then have the commutative diagram

$$\begin{array}{ccc}
\mathcal{H}_W(a_*^c) & \xrightarrow{\mathcal{F}} & \mathcal{F}_0 \\
D^i(G) \xrightarrow{A} & & \mathcal{D}_W(A)
\end{array}$$

from Ch. IV, Theorem 7.1 and Cor. 7.4.

The Schwartz space $S(a^*)$ is topologized by the seminorms

$$\tau_{P,m}(h) = \sup_{a^*} (|\lambda| + 1)^m \left| P \left( \frac{\partial}{\partial \lambda} \right) h(\lambda) \right|$$

$m \in \mathbb{Z}^+$, $P \in S(a^*)$. Since the Laplacian $L$ on $G/K$ has the property

$$L \varphi_\lambda = -((\lambda, \lambda) + |\rho|^2) \varphi_\lambda$$

it is sometimes convenient to use the seminorms

$$\tau_{P,m}(h) = \sup_{a^*} \left| P \left( \frac{\partial}{\partial \lambda} \right) ((\lambda, \lambda) + |\rho|^2) h(\lambda) \right|,$$

which define the same topology on $S(a^*)$.

We shall have use for the following simple result. If $f \in S(\mathbb{R}^n)$ then

$$\left| \int_{\mathbb{R}^n} f(x) \, dx \right| \leq c_n \sup_x (|x|^{n+1}|f(x)|)$$

where $c_n$ is a constant. In fact $|f(x)| \leq M(|x| + 1)^{-n-1}$ so left hand side of (2.7) is bounded by constant multiple of $M$.

The aim is now to prove that the bijection $\mathcal{F} : \mathcal{D}^i(G) \rightarrow \mathcal{H}_W(a_*^c)$ given by the Paley–Wiener theorem is bicontinuous for the topologies induced by $I^2(G)$ and $S(a^*)$. Let $S_W(a^*)$ denote the set of $W$-invariants in $S(a^*)$.
Lemma 2.1. The spherical transform \( f \to \tilde{f} \) given by (2) maps \( \mathcal{D}(G) \) continuously into \( S_W(a^*) \).

Proof:

To check the convergence of the integral

\[ (2.8) \quad \tilde{f}(\lambda) = \int_G f(g)\varphi_-(g) \, dg, \quad \lambda \in a^*, \]

we use Theorem 5.8 in Ch. I to reduce the integral to one over \( A^+ \). Here the density \( \delta \) satisfies

\[ (2.9) \quad \delta(\exp H) \leq c e^{2\rho(H)}, \quad H \in a^+, \]

for a constant \( c \). Since \( |\varphi_\lambda(g)| \leq \varphi_0(g) \) and, by (2.1), \( |f(g)| \leq \text{const} \,(|g| + 1)^{-q}\varphi_0(g) \) for each \( q \) the absolute convergence is clear from Theorem 1.1, (i). The smoothness in \( \lambda \) follows from Theorem 1.1 (iv). For the remaining statements we just have to prove that a seminorm \( \tau = \tau_{P,m} \) on \( S_W(a^*) \), there exists a seminorm \( \sigma = \sigma_{D,q} \) such that

\[ (2.10) \quad \tau(\tilde{f}) \leq c_1 \sigma(f) \quad f \in \mathcal{D}(G), \]

where \( c_1 \) is a constant. We have

\[ P \left( \frac{\partial}{\partial \lambda} \right) (\langle \lambda, \lambda \rangle)^m \tilde{f}(\lambda) = \int_G (-L)^m f(g) P \left( \frac{\partial}{\partial \lambda} \right) \varphi_-(g) \, dg. \]

Again we reduce the integral to \( A^+ \), use the estimates Theorem 1.1, (iv), (i), and combine with (2.9) and the estimate

\[ |(Df)(a)| \leq (|a| + 1)^{-q}\varphi_0(a) \leq c(|a| + 1)^{-q+d}e^{-\rho(\log a)}. \]

Taking \( D = (-L)^m \) and \( q \) large (2.10) follows.

We now come to Anker’s principal lemma.

Lemma 2.2. The inverse map \( \mathcal{F}^{-1} : \mathcal{H}(a^*_+ \to \mathbb{D}^2(G) \) given by

\[ (\mathcal{F}^{-1}h)(g) = f(g) = \int_{a^*} h(\lambda)\varphi_\lambda(g)|c(\lambda)|^{-2} \, d\lambda \]

is continuous in the topologies induced from \( S_W(a^*) \) and \( \mathcal{D}^2(G) \).

Proof: Given a seminorm \( \sigma = \sigma_{D,q} \) on \( \mathcal{D}^2(G) \) the problem is to find a seminorm \( \tau = \tau_{P,m} \) on \( S_W(a^*) \) such that

\[ (2.11) \quad \sigma(f) \leq \tau(h) \quad \text{for } f \in \mathcal{D}^2(G). \]

We have for \( D \in \mathcal{D}(G) \)

\[ (2.12) \quad (Df)(g) = \int_{a^*} h(\lambda)D\varphi_\lambda(g)|c(\lambda)|^{-2} \, d\lambda \]

and wish to estimate

\[ (2.13) \quad F(g) = (|g| + 1)^q \varphi_0(g)(Df)(g) \]

because \( \sigma_{D,q}(f) = \sup_g |F(g)|. \) From Theorem 1.1(iii) and Ch. IV, Prop. 7.2 we have for a suitable \( m_0 \)

\[ (2.14) \quad |(F(g))| \leq c_0(|g| + 1)^q \int_{a^*} (|\lambda| + 1)^{m_0}|h(\lambda)| \, d\lambda. \]

One would now like to remove the factor \( (|g| + 1)^q \) by replacing \( h \) with a suitable derivative. It seems hard to do this globally, that is on all of \( G \). Following Anker we do this locally, that is by dividing \( G \) up into pieces on which this process works.
Consider the balls $B_j = \{ H \in a : |H| \leq j \}, j \in \mathbb{Z}^+$, and put $G_j = K \exp B_j K$. Let $\omega \in C_\infty (\mathbb{R})$ be an even function, $0 \leq \omega (x) \leq 1$, with the properties:

$$\omega (x) = 1 \text{ for } |x| \leq \frac{1}{2}, \omega \text{ has support in } (-1, 1).$$

We define $\omega_j \in \mathcal{D}_W (a)$ for $j \geq 1$ by

$$\omega_j (H) = \begin{cases} 1 & \text{ for } |H| \leq j - 1, \\ \omega(|H| - j + 1) & \text{ for } |H| > j - 1. \end{cases}$$

Then $\omega_j$ and each of its derivatives is bounded uniformly in $j$.

With $f \in \mathcal{D}^2 (G)$ let $h = \mathcal{F} f$, $g (H) = \mathcal{A} f (\exp H)$. Thus $g \in \mathcal{D}_W (a)$ and $h \in \mathcal{H} (a^*_e)$. We decompose $g = \omega_j g + (1 - \omega_j) g$, put $g_j = (1 - \omega_j) g$ and let $f_j$ and $h_j$ be the corresponding functions in $\mathcal{D}^2 (G)$ and $\mathcal{H} (a^*_e)$, respectively. We know from Ch. IV, Theorem 7.1 and Cor. 7.4, that for each closed ball $B \subset a$ with center 0,

$${\text{supp}} (f) \subset K \exp B K \iff {\text{supp}} (g) \subset B.$$ 

Since $g - g_j = 0$ outside $B_j$, $f - f_j = 0$ outside $G_j$. The constants below will depend on $\sigma$ but neither on $f$ nor on $j$. We now use (2.12) with $f$ and $h$ replaced by $f_j$ and $h_j$, respectively. This does not change $F$ outside $G_j$. Thus using (2.7) (and $j + 2 \leq 3j$) (2.14) implies

$$\sup_{G_{j+1} \setminus G_j} |F (g)| \leq c_1 j^q \tau_{1,m} (h_j), \quad m = m_0 + \dim a + 1.$$ 

We shall now prove, by Euclidean Fourier analysis, that given $q, m \in \mathbb{Z}^+$ there exists a seminorm

$$\tau_{d,t}^* (h) = \sum_{k=0}^{d} \sup_{a^*_e} (|\lambda| + 1)^{l} |\nabla^k h (\lambda)| \quad (\nabla = \text{gradient})$$

such that for all $j$,

$$j^q \tau_{1,m} (h_j) \leq c_2 \tau_{d,t}^* (h).$$

This would prove (2.11). For this consider

$$h_j (\lambda) = \int_a g_j (H) e^{-i \lambda (H)} dH.$$ 

We now shift polynomial factors on $h_j$ to derivatives of $g_j$. Using (2.7) and the fact that $g_j$ vanishes on $B_{j-1}$ we get with $p = q + \dim a + 1$,

$$j^q \tau_{1,m} (h_j) \leq j^q c_3 \sum_{k=0}^{m} \int_a |\nabla^k g_j (H)| dH$$

$$\leq c_4 \sum_{k=0}^{m} \sup_{a} (|H| + 1)^{p} |\nabla^k g_j (H)|$$

$$\leq c_5 \sum_{k=0}^{m} \sup_{a} (|H| + 1)^{p} |\nabla^k g (H)|.$$
the last inequality coming from calculating the derivatives of $g_j = (1 - \omega_j)q$ by the product rule and recalling that each derivative of $1 - \omega_j$ is uniformly bounded in $j$. On the other hand,

$$g(H) = c_6 \int_{a^*} h(\lambda)e^{i\lambda(H)} d\lambda,$$

so from the last inequality we derive

$$j^q\tau_{1,m}(h_j) \leq c_7 \sum_{\ell=0}^{p} \int_{a^*} (|\lambda| + 1)^m |\nabla^\ell h(\lambda)| d\lambda$$

which again by (2.7) is dominated by a suitable $\tau_{p,m}^*(h)$. This proves the lemma.

It is well known that $\mathcal{D}(R^n)$ is dense in $S(R^n)$. This implies for our situation that $\mathcal{H}_W(a^*_1)$ is dense in $S_W(a^*)$.

**Lemma 2.3.** $\mathcal{D}^2(G)$ is dense in $\mathcal{I}^2(G)$.

**Proof:** We extend the function $\omega_j$ above to a $K$-bi-invariant smooth function $\psi_j$ on $G$. Then $\psi_j \equiv 1$ on $G_{j-1}$ and $\psi_j = 0$ outside $G_j$. Consider the seminorm $\sigma = \sigma_{D,q}$ in (2.1). Then

$$\sigma(\psi_j f - f) \leq \sup_{|g| > j-1} (1 + |g|)^q \varphi_0(g)^{-1} |D(\psi_j f - f)(g)|$$

$$\leq \frac{1}{j^q} (\sigma_{D,q+1}(\psi_j f) + \sigma_{D,q+1}(f)).$$

Also

$$D(\psi_j f) = \sum_i D_i(\psi_j) E_i(f) \quad D_i, E_i \in \mathcal{D}(G)$$

so our expression is majorized by

$$c \frac{1}{j^q} \sum_i (\sigma_{E_i,q+1}(f) + \sigma_{D,q+1}(f))$$

where $c$ is a constant. Here we used again the uniform boundedness of $D_i\psi_i$ in $j$. The last estimate shows $\sigma(\psi_j f - f) \to 0$ proving the lemma.

**Theorem 2.4.** The spherical transform $\mathcal{F} : f \to \tilde{f}$ given by

$$\tilde{f}(\lambda) = \int_G f(g)\varphi_{-\lambda}(g) dg$$

is a homeomorphism of $\mathcal{I}^2(G)$ onto $S_W(a^*)$. The inverse $\mathcal{F}^{-1}$ is given by

$$(\mathcal{F}^{-1}h)(g) = \text{const} \int_{a^*} h(\lambda)\varphi_\lambda(g)|c(\lambda)|^{-2} d\lambda.$$

**Proof:** The spaces $\mathcal{I}^2(G)$ and $S_W(a^*)$ are Fréchet spaces. Because of their completeness and the density in Lemma 2.3 the inverse $\mathcal{F}^{-1}$ extends to a linear homeomorphism of $S_W(a^*)$ onto $\mathcal{I}^2(G)$. The inverse of this map must by Lemma 2.2 coincide with $\mathcal{F}$. Thus $\mathcal{F} : \mathcal{I}^2(G) \to S_W(a^*)$ is surjective.

We must still prove that our “abstract” extension of $\mathcal{F}^{-1}$ to $S_W(a^*)$ is given by (2.15). Let $h \in S_W(a^*)$ and let $h_n \in \mathcal{H}_W(a^*_7)$ converge to $h$. Since $\varphi_\lambda(g)$ is bounded, and $|c(\lambda)|^{-2}$ bounded by a polynomial, we have $\tau_{1,m}(h_n - h) \to 0$. Thus the validity of (2.15) for $h_n$ implies its validity for $h$.  

6
Reference