Some Personal Remarks on the Radon Transform.

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1. Introduction.

The editors have kindly asked me to write here a personal account of some of my work concerning the Radon transform. My interest in the subject was actually evoked during a train trip from New York to Boston once during the Spring 1955.

2. Some old times.

Back in 1955, I worked on extending the mean value theorem of Leifur Ásgeirsson [1937] for the ultrahyperbolic equation on $\mathbb{R}^n \times \mathbb{R}^n$ to Riemannian homogeneous spaces $G/K \times G/K$. I was motivated by Godement’s generalization [1952] of the mean value theorem for Laplace’s equation $Lu = 0$ to the system $Du = 0$ for all $G$-invariant differential operators $D$ (annihilating the constants) on $G/K$. At the time (Spring 1955) I visited Leifur in New Rochelle where he was living in the house of Fritz John (then on leave from NYU). They had both been students of Courant in Göttingen in the 1930’s. Since John’s book [1955] treats Ásgeirsson’s theorem in some detail, Leifur lent me a copy of it (in page proofs) to look through on the train to Boston.

I was quickly enticed by Radon’s formulas (in John’s formulation) for a function $f$ on $\mathbb{R}^n$ in terms of its integrals over hyperplanes. In John’s notation, let $J(\omega, p)$ denote the integral of $f$ over the hyperplane...
\( \langle \omega, x \rangle = p \ (p \in \mathbb{R}, \ \omega \ \text{a unit vector}), \ d\omega \ \text{the area element on } S^{n-1} \ \text{and} \ L \ \text{the Laplacian.} \Then \\
\begin{align*}
(2.1) \ f(x) &= \frac{1}{2} (2\pi i)^{1-n} (L_x)^{(n-1)/2} \int_{S^{n-1}} J(\omega, \langle \omega, x \rangle) \ d\omega, \ n \ \text{odd.} \\
(2.2) \ f(x) &= (2\pi i)^{-n} (L_x)^{(n-2)/2} \int_{S^{n-1}} d\omega \int_{\mathbb{R}} \frac{dJ(\omega, p)}{p - \langle \omega, x \rangle}, \ n \ \text{even.}
\end{align*}

I was surprised at never having seen such formulas before. Radon’s paper [1917] was very little known, being published in a journal that was hard to find. The paper includes some suggestions by Herglotz in Leipzig and John learned of it from lectures by Herglotz in Göttingen. I did not see Radon’s paper until several years after the appearance of John’s book but it has now been reproduced in several books about the Radon transform (terminology introduced by F. John). Actually, the paper is closely related to earlier papers by P. Funk [1913, 1916] (quoted in Radon [1917]) which deal with functions on \( S^2 \) in terms of their integrals over great circles. Funk’s papers are in turn related to a paper by Minkowski [1911] about surfaces of constant width.

Considering formula (2.1) for \( \mathbb{R}^3 \),
\[(3.1) \ f(x) = -\frac{1}{8\pi^2} L_x \left( \int_{S^2} J(\omega, \langle \omega, x \rangle) \ d\omega \right) \]
it struck me that the formula involves two dual integrations, \( J \) the integral over the set of points in a plane and then \( d\omega \), the integral over the set of planes through a point. This suggested the operators \( f \to \hat{f}, \varphi \to \varphi ^{\vee} \) defined as follows:

For functions \( f \) on \( \mathbb{R}^3 \), \( \varphi \) on \( \mathbb{P}^3 \) (the space of 2-planes in \( \mathbb{R}^3 \)) put
\[(3.2) \ \hat{f}(\xi) = \int f(x) \ dm(x), \quad \xi \in \mathbb{P}^3, \]
\[(3.3) \ \varphi ^{\vee}(x) = \int \varphi(\xi) \ d\mu(\xi), \quad x \in \mathbb{R}^3, \]
where \( dm \) is the Lebesgue measure and \( d\mu \) the average over all hyperplanes containing \( x \). Then (3.1) can be rewritten
\[(3.4) \ f = -\frac{1}{2} L((\hat{f})^{\vee}). \]
The spaces $\mathbb{R}^3$ and $\mathbb{P}^3$ are homogeneous spaces of the same group $\mathrm{M}(3)$, the group of isometries of $\mathbb{R}^3$, in fact
\[ \mathbb{R}^3 = \mathrm{M}(3)/\mathrm{O}(3), \quad \mathbb{P}^3 = \mathrm{M}(3)/\mathbb{Z}_2\mathrm{M}(2). \]
The operators $f \to \hat{f}$, $\varphi \to \varphi^\vee$ in (3.2)–(3.3) now generalize ([1965a], [1966]) to homogeneous spaces
\[ X = G/K, \quad \Xi = G/H, \]
f and $\varphi$ being functions on $X$ and $\Xi$, respectively, by
\[ \hat{f}(\gamma H) = \int_{H/L} f(\gamma hK) dh_L, \quad \varphi^\vee(gK) = \int_{K/L} \varphi(gkH) dk_L. \]
Here $G$ is an arbitrary locally compact group, $K$ and $H$ closed subgroups, $L = K \cap H$, and $dh_L$ and $dk_L$ the essentially unique invariant measure on $H/L$ and $K/L$, respectively. This is the abstract Radon transform for the double fibration:
\[ X = G/K \quad \Xi = G/H \]

The operators $f \to \hat{f}$, $\varphi \to \varphi^\vee$ map functions on $X$ to functions on $\Xi$ and vice-versa. These geometrically dual operators also turned out to be adjoint operators relative to the invariant measures $dx$ and $d\xi$,
\[ \int_X f(x)\varphi^\vee(x) dx = \int_{\Xi} \hat{f}(\xi)\varphi(\xi) d\xi. \]
This suggests natural extensions to suitable distributions $T$ on $X$, $\Sigma$ on $\Xi$, as follows:
\[ \hat{T}(\varphi) = T(\varphi^\vee), \quad \Sigma(\hat{f}) = \Sigma(f^\vee). \]

Formulas (2.1) and (2.2) have another interesting feature. As functions of $x$ the integrands are plane waves, i.e. constant on each hyperplane $L$ perpendicular to $\omega$. Such a function is really just a function of a single variable so (2.1) and (2.2) can be viewed as a decomposition of an $n$-variable function into one-variable functions. This feature enters into the work of Herglotz and John [1955]. I have found some applications of an analog of this principle for invariant differential equations on symmetric spaces ([1963], §7, [2008], Ch. V§1, No. 4), where parallel planes are replaced by parallel horocycles.
The setup (3.5) and (3.6) above has of course an unlimited supply of examples. Funk’s example
\[(3.9) \quad X = S^2, \quad \Xi = \{\text{great circles on } S^2\}\]
fits in, both X and Ξ being homogeneous spaces of O(3).

Note that with given X and Ξ there are several choices for K and H. For example, if \(X = \mathbb{R}^n\) we can take K and H, respectively, as the isotropy groups of the origin O and a k-plane ξ at distance p from O. Then the second transform in (3.6) becomes
\[(3.10) \quad \varphi_p^\vee(x) = \int_{d(x, \xi) = p} \varphi(\xi) d\mu(\xi)\]
and we get another inversion formula (cf. [1990], [2011]) of \(f \to \hat{f}\) involving \((\hat{f})_p^\vee(x)\), different from (3.4).

Similarly, for X and Ξ in (3.9) we can take K as the isotropy group of the North Pole N and H as the isotropy group of a great circle at distance p from N. Then \((\hat{f})_p^\vee(x)\) is the average of the integrals of f over the geodesics at distance p from x.

The principal problems for the operators \(f \to \hat{f}, \varphi \to \varphi^\vee\) would be
A. Injectivity.
B. Inversion formulas.
C. Ranges and kernels for specific function spaces of X and on Ξ.
D. Support problems (does compact support of \(\hat{f}\) imply compact support of \(f\)?)

These problems are treated for a number of old and new examples in [2011]. Some unexpected analogies emerge, for example a complete parallel between the Poisson integral in the unit disk and the X-ray transform in \(\mathbb{R}^3\); see pp. 86–89, loc. cit.

My first example for (3.5) and (3.6) was for X a Riemannian manifold of constant sectional curvature c and dimension n and Ξ the set of k-dimensional totally geodesic submanifolds of X. The solution to Problem B is then given [1959] by the following result, analogous to (3.4):

For k even let \(Q_k\) denote the polynomial
\[Q_k(x) = [x-c(k-1)(n-k)][x-c(k-3)(n-k+2)] \cdots [x-c(1)(n-2)].\]
Then for a constant γ
\[(3.11) \quad Q_k(L)((\hat{f})^\vee) = \gamma f \quad \text{if } X \text{ is noncompact.}\]
\[(3.12) \quad Q_k(L)((\hat{f})^\vee) = \gamma(f + f \circ A) \quad \text{if } X \text{ is compact.}\]
In the latter case $X = S^n$ and $A$ the antipodal mapping. The constant $\gamma$ is given by
\[ \Gamma \left( \frac{n}{2} \right) / \Gamma \left( \frac{n-k}{2} \right) (-4\pi)^{k/2} . \]

The proof used the generalization of Ásgeirsson’s theorem. At that time I had also proved an inversion formula for $k$ odd and the method used in the proof of the support theorem for $H^n$ ([1964, Theorem 5.2]) but not published until [1990] since the formula seemed unreasonably complicated in comparison to (3.11), (3.12) and since the case $k = 1$ when $f \to \hat{f}$ is the "X-ray transform", had not reached its distinction through tomography.

For $k$ odd the inversion formula is a combination of $\hat{f}$ and the analog of (3.10).

For $X = R^n$, $\Xi = P^n$, problems C–D are dealt with in [1965a]. This paper also solves problem B for $X$ any compact two-point homogeneous space and $\Xi$ the family of antipodal submanifolds.

4. Horocycle duality.

In the search of a Plancherel formula for simple Lie groups, Gelfand–Naimark [1957], Gelfand–Graev [1955] and Harish-Chandra [1954], [1957] showed that a function of $f$ on $G$ is explicitly determined by the integrals of $f$ over translates of conjugacy classes in $G$. This did not fit into the framework (3.5)(3.6) so using the Iwasawa decomposition $G = KAN$ ($K$ compact, $A$ abelian, $N$ nilpotent) I replaced the conjugacy classes by their “projections” in the symmetric space $G/K$, and this leads to the orbits of the conjugates $gNg^{-1}$ in $G/K$. These orbits are the horocycles in $G/K$. They occur in classical non-Euclidean geometry (where they carry a flat metric) and for $G$ complex are extensively discussed in Gelfand–Graev [1964].

For a general semisimple $G$, the action of $G$ on the symmetric space $G/K$ turned out to permute the horocycle transitively with isotropy group $MN$, where $M$ is the centralizer of $A$ in $K$ [1963]. This leads (3.5) and (3.6) to the double fibration

\[ X = G/K \quad \Xi = G/MN \]

and for functions $f$ on $X$, $\varphi$ on $\Xi$, $\hat{f}(\xi)$ is the integral of $f$ over a horocycle $\xi$ and $\varphi(x)$ is the average of $\varphi$ over the set of horocycles through $x$. My
papers [1963], [1964a], [1970] are devoted to a geometric examination of this duality and its implications for analysis, differential equations and representation theory. Thus we have double coset space representations

\[ K \backslash G / K \approx A / W, \quad MN \backslash G / MN \approx W \times A \]

based on the Cartan and Bruhat decomposition of \( G \), \( W \) denoting the Weyl group.

The finite-dimensional irreducible representations with a \( K \)-fixed vector turn out to be the same as those with an \( MN \)-fixed vector. This leads to simultaneous imbeddings of \( X \) and \( \Xi \) into the same vector space and the horocycles are certain plane sections with \( X \) in analogy with their flatness for \( H^m \) [2008,II,§4]. The set of highest restricted weights of these representations is the dual of the lattice \( \Sigma_i \mathbb{Z}^+ \beta_i \) where \( \beta_1, \ldots, \beta_\ell \) is the basis of the unmultipliable positive restricted roots.

For the algebras \( D(X), D(\Xi) \), respectively \( D(A) \), of \( G \)-invariant (resp. \( A \)-invariant) differential operators on \( X, \Xi \) and \( A \) we have the isomorphisms

\[ D(X) \approx D(A) / W, \quad D(\Xi) \approx D(A) . \]

The first is a reformulation of Harish-Chandra’s homomorphism, the second comes from the fact that the \( G \)-action on the fibration of \( \Xi \) over \( K / M \) is fiber preserving and generates a translation on each (vector) fiber. The transforms \( f \to \tilde{f}, \varphi \to \varphi \) intertwine the members of \( D(X) \) and \( D(\Xi) \). In particular, when the operator \( f \to \tilde{f} \) is specialized to \( K \)-invariant functions on \( X \) it furnishes a simultaneous transmutation operator between \( D(X) \) and the set of \( W \)-invariants in \( D(A) \) [1964a, §2]. This property, combined with a surjectivity result of Hörmander [1958] and Lojasiewicz [1958] for tempered distributions on \( \mathbb{R}^n \), yields the result that each \( D \in D(X) \) has a fundamental solution [1964a]. A more technical support theorem in [1973] for \( f \to \tilde{f} \) on \( G / K \) then implies the existence theorem that each \( D \in D(X) \) is surjective on \( \mathcal{E}(X) \) i.e. \( (\mathcal{E}^\infty(X)) \). It is also surjective on the space \( \mathcal{D}'(X) \) of \( K \)-finite distributions on \( X \) ([1976]) and on the space \( \mathcal{S}'(X) \) of tempered distributions on \( X \) ([1973a]). The surjectivity on all of \( \mathcal{D}'(X) \) however seems as yet unproved.

The method of [1973] also leads to a Paley–Wiener type Theorem for the horocycle transform \( f \to \tilde{f} \), that is an internal description of the range \( \mathcal{D}(X) \). The formulation is quite different from the analogous result for \( \mathcal{D}(\mathbb{R}^n) \). Having proved the latter result in the summer of 1963, I always remember when I presented it in a Fall class, because immediately afterwards I heard about John Kennedy’s assassination.
In analogy with (3.4), (3.11), the horocycle transform has an inversion formula. The parity difference in (2.1), (2.2) and (3.11), (3.12) now takes another form ([1964], [1965b]):

If $G$ has all its Cartan subgroups conjugate then

$$f = \Box((\hat{f})^\vee),$$

where $\Box$ is an explicit operator in $D(X)$. Although this remains “formally valid” for general $G$ with $\Box$ replaced by a certain pseudo-differential operator, a better form is

$$f = (\Lambda \hat{f})^\vee,$$

with $\Lambda$ a certain pseudo-differential operator on $\Xi$. These operators are constructed from Harish-Chandra’s $c$-function for $G$. For $G$ complex a formula related to (4.5) is stated in Gelfand and Graev [1964], §5.

As mentioned, $f \to \hat{f}$ is injective and the range $D(X)^\vee$ is explicitly determined. On the other hand $\varphi \to \varphi^\vee$ is surjective from $C^\infty(\Xi)$ onto $C^\infty(X)$ but has a big describable kernel.

By definition, the spherical functions on $X$ are the $K$-invariant eigenfunctions of the operators in $D(X)$. By analogy we define conical distribution on $\Xi$ to be the $MN$-invariant eigendistributions of the operators in $D(\Xi)$. While Harish-Chandra’s formula for the spherical functions parametrizes the set $\mathcal{F}$ of spherical functions by

$$\mathcal{F} \approx a^*_c / W$$

(where $a^*_c$ is the complex dual of the Lie algebra of $A$) the space $\Phi$ of conical distribution is “essentially” parametrized by

$$\Phi \approx a^*_c \times W.$$

Note the analogy of (4.6), (4.7), with (4.2). In more detail, the spherical functions are given by

$$\varphi_\lambda(gK) = \int_K e^{i(\lambda - \rho)(H(gk))} dk,$$

where $g = k \exp H(g)n$ in the Iwasawa decomposition, $\rho$ and $\lambda$ as in (5.3) and $\lambda$ unique $\mod W$. On the other hand, the action of the group $MNA$ on $\Xi$ divides it into $|W|$ orbits $\Xi_\lambda$ and for $\lambda \in a^*_c$ a conical distribution is constructed with support in the closure of $\Xi_\lambda$. The construction is done by a specific holomorphic continuation. The identification in (4.7) from [1970] is complete except for certain singular eigenvalues. For the case of $G/K$ of rank one the full identification of (4.7) was completed by Men-Cheng Hu in his MIT thesis [1973]. Operating as convolutions on $K/M$ the conical distributions in (4.7) furnish intertwining operators
in the spherical principal series [1970, Ch. III, Theorem 6.1]. See also Schiffmann [1971], Théorème 2.4 and Knapp-Stein [1971].

5. A Fourier transform on $X$.

Writing $\tilde{f}(\omega, p)$ for John’s $J(\omega, p)$ in (2.1) the Fourier transform $\tilde{f}$ on $\mathbb{R}^n$ can be written

$$\tilde{f}(r\omega) = \int_{\mathbb{R}^n} f(x)e^{-ir\langle x, \omega \rangle} \, dx = \int_{\mathbb{R}} \hat{f}(\omega, p)e^{-irp} \, dp,$$

which is the one-dimensional Fourier transform of the Radon transform. The horocycle duality would call for an analogous Fourier transform on $X$.

The standard representation-theoretic Fourier transform on $G$,

$$\tilde{F}(\pi) = \int_{G} F(x)\pi(x) \, dx$$

is unsuitable here because it assigns to $F$ a family of operators in different Hilbert spaces. However, the inner product $\langle x, \omega \rangle$ in (5.1) has a certain vector–valued analog for $G/K$, namely

$$A(gK, kM) = A(k^{-1}g),$$

where $\exp A(g)$ is the $A$-component in the Iwasawa decomposition $G = NAK$. Writing for $x \in X$, $b \in B = K/M$,

$$e_{\lambda,b}(x) = e^{(i\lambda + \rho)(A(x,b))}, \quad \lambda \in \mathfrak{a}^*, \quad \rho(H) = \frac{1}{2} \text{Tr} (\text{ad} \, H|n)$$

we define in [1965b] a Fourier transform,

$$\tilde{f}(\lambda, b) = \int_{X} f(x)e_{-\lambda,b}(x) \, dx.$$

The analog of (5.1) is then

$$\tilde{f}(\lambda, kM) = \int_{A} \hat{f}(kaMN)e^{(-i\lambda + \rho)(\text{log} \, a)} \, da.$$}

The main theorems of the Fourier transform on $\mathbb{R}^n$, the inversion formula, the Plancherel theorem (with range), the Paley–Wiener theorem, the Riemann–Lebesgue lemma, have analogs for this transform ([1965b], [1973], [2005], [2008]). The inversion formula is based on the new identity

$$\varphi_{\lambda}(g^{-1}h) = \int_{K} e^{(i\lambda + \rho)(A(kh))} e^{(-i\lambda + \rho)(A(kg))} \, dk.$$
Some results have richer variations, like the range theorems for the various Schwartz spaces $S_p(X) \subset L^p(X)$ (Eguchi [1979]).

The analog of (5.4) for the compact dual symmetric space $U/K$ was developed by Sherman [1977], [1990] on the basis of (5.5).


The Harish-Chandra formula for spherical functions can be written in the form

\begin{equation}
\varphi_{\lambda}(x) = \int_B e^{i(\lambda + \rho)(A(x,b))} \, db
\end{equation}

with $\lambda \in \mathfrak{a}^*_c$ given mod $W$-invariance. These are the $K$-invariant joint eigenfunctions of the algebra $D(X)$. The spaces

\begin{equation}
\mathcal{E}_\lambda(X) = \{ f \in \mathcal{E}(X) : \int_K f(gk \cdot x) \, dk = f(g \cdot o)\varphi_{\lambda}(x) \}
\end{equation}

were in [1962] characterized as the joint eigenspaces of the algebra $D(X)$. Let $T_{\lambda}$ denote the natural representation of $G$ on $\mathcal{E}_\lambda(X)$. Similarly, for $\lambda \in \mathfrak{a}^*_c$ the space $\mathcal{D}'_{\lambda}(\Xi)$ of distributions $\Psi$ on $\Xi$ given by

\begin{equation}
\Psi(\varphi) = \int_{K/M} \left( \int_A \varphi(kaMN)e^{i(\lambda + \rho)(\log a)} \, da \right) dS(kM)
\end{equation}

is the general joint distribution eigenspace for the algebra $D(\Xi)$. Here $S$ runs through all of $\mathcal{D}'(B)$. The dual map $\Psi \mapsto \Psi^*$ maps $\mathcal{D}'_{\lambda}(\Xi)$ into $\mathcal{E}_\lambda(X)$. In terms of $S$ this dual mapping amounts to the Poisson transform $P_\lambda$ given by

\begin{equation}
P_\lambda S(x) = \int_B e^{i(\lambda + \rho)(A(x,b))} \, dS(b).
\end{equation}

By definition the Gamma function of $X$, $\Gamma_X(\lambda)$, is the denominator in the formula for $c(\lambda)c(-\lambda)$ where $c(\lambda)$ is the $c$-function of Harish-Chandra, Gindikin and Karpelevič. While $\Gamma_X(\lambda)$ is a product over all indivisible roots, $\Gamma_X^+(\lambda)$ is the product over just the positive ones. See [2008], p. 284. In [1976] it is proved that for $\lambda \in \mathfrak{a}^*_c$,

(i) $T_{\lambda}$ is irreducible if and only if $1/\Gamma_X(\lambda) \neq 0$.

(ii) $P_{\lambda}$ is injective if and only if $1/\Gamma_X^+(\lambda) \neq 0$.

(iii) Each $K$-finite joint eigenfunction of $D(X)$ has the form
\[ (6.5) \quad \int_B e^{(i\lambda + \rho)(A(x,b))} F(b) \, db \]

for some \( \lambda \in a_+^* \) and some \( K \)-finite function \( F \) on \( B \).

For \( X = H^n \) it was shown [1970, p.139, 1973b] that all eigenfunctions of the Laplacian have the form (6.5) with \( F(b) \, db \) replaced by an \textit{analytic functional} (hyperfunction). This was a bit of a surprise since this concept was in very little use at the time. The proof yielded the same result for all \( X \) of rank one provided eigenvalue is \( \geq -\langle \rho, \rho \rangle \). In particular, all harmonic functions on \( X \) have the form
\[ (6.6) \quad u(x) = \int_B e^{2\rho(A(x,b))} \, dS(b), \]
where \( S \) is a hyperfunction on \( B \).

For \( X \) of arbitrary rank it was proved by Kashiwara, Kowata, Minomura, Okamoto, Oshima and Tanaka that \( P_\lambda \) is \textit{surjective} for \( 1/\Gamma_X^+(\lambda) \neq 0 \) [1978]. In particular, every joint eigenfunction has the form (6.4) for a suitable hyperfunction \( S \) on \( B \). The image under \( P_\lambda \) of various other spaces on \( B \) has been widely investigated, we just mention Furstenberg [1963], Karpelevič [1963], Lewis [1978], Oshima–Sekiguchi [1980], Wallach [1983], Ban–Schlichtkrull [1987], Okamoto [1971], Yang [1998].

For the compact dual symmetric space \( U/K \) the eigenspace representations are all irreducible and each joint eigenfunction is of the form (6.5) (cf. Helgason [1984] Ch. V, §4, in particular p. 542). Again this relies on (5.5).

7. The X-ray transform.

The \textit{X-ray transform} \( f \rightarrow \hat{f} \) on a complete Riemannian manifold \( X \) is given by
\[ (7.1) \quad \hat{f}(\gamma) = \int_\gamma f(x) \, dm(x), \quad \gamma \text{ a geodesic}, \]
\( f \) being a function on \( X \). For the symmetric space \( X = G/K \) from §4, I showed in [1980] the injectivity and support theorem for (7.1) (problems A and D in §3). In [2006], Rouvière proved an explicit inversion formula for (7.1).

For a compact symmetric space \( X = U/K \) we assume \( X \) irreducible and simply connected. Here we modify (7.1) by restricting \( \gamma \) to be a closed geodesic of minimal length, and call the transform the \textit{Funk transform}. All such geodesics are conjugate under \( U \) (Helgason [1966a] so the family \( \Xi = \{ \gamma \} \) has the form \( U/H \) and the Funk transform falls
in the framework (3.7). The injectivity (for $X \neq S^n$) was proved by Klein, Thorbergsson and Verhóczki [2009]; an inversion formula and a support theorem by the author [2007]. To each $x \in X$ is associated the midpoint locus $A_x$ (the set of midpoints of minimal geodesics through $x$) as well as a corresponding “equator” $E_x$. Both of these are acted on transitively by the isotropy group of $x$. The inversion formula involves integrals over both $A_x$ and $E_x$.

For a closed subgroup $H \subset G$, invariant under the Cartan involution $\theta$ of $G$ (with fixed group $K$) Ishikawa [2003] investigated the double fibration (3.7). The orbit $HK$ is a totally geodesic submanifold of $X$ so this generalizes the X-ray transform. For many cases of $H$, this new transform was found to be injective and to satisfy a support theorem.

For one variation of these questions see Frigyik, Stefanov and Uhlmann [2008].

8. Concluding remarks.

For the sake of unity and coherence, the account in the sections above has been rather narrow and group-theory oriented. A satisfactory account of progress on Problems A, B, C, D in §3 would be rather overwhelming. My book [2011] with its bibliographic notes and references is a modest attempt in this direction.

Here I restrict myself to the listing of topics in the field — followed by a bibliography, hoping the titles will serve as a suggestive guide to the literature. Some representative samples are mentioned. These samples are just meant to be suggestive, but I must apologize for the limited exhaustiveness.

(i) **Topological properties of the Radon transform.** Quinto [1981], Hertle [1984a].

(ii) **Range questions for a variety of examples of $X$ and $\Xi$.** The first paper in this category is John [1938] treating the X-ray transform in $\mathbb{R}^3$. For $X$ the set of $k$-planes in $\mathbb{R}^n$ the final version, following intermediary results by Helgason [1980b], Gelfand, Gindikin and Graev [1982], Richter [1986], [1990], Kurusa [1991], is in Gonzalez [1990b] where the range is the null space of an explicit $4^{th}$ degree differential operator. Enormous progress has been made for many examples. Remarkable analogies have emerged, Berenstein, Kurusa, Casadio Tarabusi [1997]. For Grassmann manifolds and spheres see e.g. Kakehi [1993], Gonzalez and Kakehi [2004]. Also Oshima [1996], Ishikawa [1997].


(vi) Extensions to forms and vector bundles. Okamoto [1971], Goldschmidt [1990].


Hopefully the titles in the following bibliography will furnish helpful contact with topics listed above.
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