1. **4.1.1 (a), (b), (d), 4.1.2.**

**Solution. 4.1.1 (a)**

\[
f(x) = \sin^3 x = \sin x \cdot \sin^2 x = \sin x \cdot \frac{1 - \cos 2x}{2}
\]

\[
= \frac{1}{2} \sin x - \frac{1}{2} \sin x \cos 2x = \frac{1}{2} \sin x - \frac{1}{4} (\sin 3x - \sin x)
\]

\[
= \frac{3}{4} \sin x - \frac{1}{4} \sin 3x.
\]

(b) Since \( f(x) = |\sin x| \) is even, it’s Fourier series is of the form

\[
f(x) = |\sin x| = a_0 + \sum_{k=1}^{\infty} a_k \cos kx,
\]

where

\[
a_0 = \frac{1}{\pi} \int_0^{\pi} \sin x \, dx = \frac{1}{\pi} (-\cos x)|_0^\pi = \frac{2}{\pi},
\]

and

\[
a_k = \frac{2}{\pi} \int_0^{\pi} \sin x \cos kx \, dx = \frac{2}{\pi} \int_0^{\pi} \frac{\sin (k+1)x - \sin (k-1)x}{2} \, dx
\]

\[
= \frac{1}{\pi} (-\cos \frac{(k+1)x}{k+1} + \cos \frac{(k-1)x}{k-1}) = \frac{1 + (-1)^k}{\pi} \left( \frac{1}{k-1} - \frac{1}{k+1} \right)
\]

\[
= \begin{cases} 
-\frac{4}{\pi(k^2-1)} & \text{if } k \text{ is even}, \\
0 & \text{if } k \text{ is odd}.
\end{cases}
\]

So the Fourier series is

\[
f(x) = |\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}.
\]

(d) The complex form of the Fourier series is

\[
f(x) = e^x = \sum_{k=-\infty}^{\infty} c_k e^{ikx},
\]

where

\[
c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-ikx} \, dx = \frac{1}{2\pi} \left( \frac{e^{(1-ik)x}}{1-ik} \right)|_{-\pi}^{\pi} = (-1)^k \frac{e^{\pi} - e^{-\pi}}{2\pi(1-ik)}.
\]
So the Fourier series is
\[ f(x) = e^x = \frac{e^\pi - e^{-\pi}}{2\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{1 - ik} e^{ikx}. \]

4.1.2. (1) Since \( f(x) \) is odd, all the cosine coefficients \( a_k \) vanish.

(2) \[
b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx = \frac{1}{\pi} \left( \int_{0}^{\pi} f(x) \sin kx \, dx - \int_{-\pi}^{0} f(x) \sin kx \, dx \right)
= \frac{1}{\pi} \left[ \left( -\frac{\cos kx}{k} \right) \Big|_{0}^{\pi} - \left( -\frac{\cos kx}{k} \right) \Big|_{-\pi}^{0} \right]
= \frac{2(1 - (-1)^k)}{k\pi}
= \begin{cases} 
0 & \text{if } k \text{ is even}, \\
\frac{4}{k\pi} & \text{if } k \text{ is odd}.
\end{cases}
\]

So the Fourier sine series is
\[ f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin (2n-1)x}{2n-1}. \]

2. (a) Let \( f(x) \) be a periodic function of period \( 2\pi \) with continuous first and second order derivatives, and \( \bar{f} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx \). Show that
\[
\int_{-\pi}^{\pi} (f'(x))^2 \, dx \geq \int_{-\pi}^{\pi} (f(x) - \bar{f})^2 \, dx,
\]
and, if
\[
\int_{-\pi}^{\pi} (f'(x))^2 \, dx = \int_{-\pi}^{\pi} (f(x) - \bar{f})^2 \, dx,
\]
then \( f(x) \) is of the form \( f(x) = a + b \cos x + c \sin x \), where \( a, b \) and \( c \) are constants.

(b) Let \( r \) be a simple closed curve on \( \mathbb{R}^2 \) of length \( 2\pi \), and \( r(s) = (x(s), y(s)) \), \(-\pi \leq s \leq \pi\), the arc length parametrization of \( r \). Assume the area inside \( r \) is \( A \). Show that
\[
A = \frac{1}{2 \pi} \int_{-\pi}^{\pi} (x(s) - \bar{x}) y'(s) \, ds,
\]
where \( \bar{x} = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(s) \, ds \).

(c) Use (a) and (b) to prove that \( A \leq \pi \), and, if \( A = \pi \), then \( r \) is a circle of radius 1. (Hint: use the facts that \( 2\pi = \int_{-\pi}^{\pi} ds \) and \( (dx/ds)^2 + (dy/ds)^2 = 1 \).)

Proof. (a) Let the Fourier series of \( f(x) \) on \([-\pi, \pi]\) be
\[
f(x) = a_0 + \sum_{n=1}^{\infty} (a_k \cos kx + b_k \sin kx).
\]
Clearly, we have \( f = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = a_0 \). So, on \([-\pi, \pi]\),
\[
f(x) - f = \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),
\]
and, hence,
\[
\int_{-\pi}^{\pi} (f(x) - f)^2 dx = \sum_{n=1}^{\infty} (a_n^2 + b_n^2).
\]
The Fourier series of \( f'(x) \) on \([-\pi, \pi]\) is
\[
f'(x) = \sum_{n=1}^{\infty} k(b_n \cos nx - a_n \sin nx).
\]
So
\[
\int_{-\pi}^{\pi} (f'(x))^2 dx = \sum_{n=1}^{\infty} k^2(a_n^2 + b_n^2),
\]
and
\[
\int_{-\pi}^{\pi} (f'(x))^2 dx - \int_{-\pi}^{\pi} (f(x) - f)^2 dx = \sum_{n=1}^{\infty} (k^2 - 1)(a_n^2 + b_n^2)
= \sum_{n=2}^{\infty} (k^2 - 1)(a_n^2 + b_n^2) \geq 0.
\]
This shows
\[
\int_{-\pi}^{\pi} (f'(x))^2 dx \geq \int_{-\pi}^{\pi} (f(x) - f)^2 dx.
\]
If
\[
\int_{-\pi}^{\pi} (f'(x))^2 dx = \int_{-\pi}^{\pi} (f(x) - f)^2 dx,
\]
then \( a_k = b_k = 0 \) for \( \forall \ k \geq 2 \), i.e., \( f(x) \) is of the form \( f(x) = a + b \cos x + c \sin x \).

(b) Let \( U \) be the region inside \( r \). Then \( A \) is the area of \( U \). By Stokes Theorem, we have
\[
\int_{-\pi}^{\pi} (x(s) - \overline{x})y'(s) \, ds = \int_{r} (x - \overline{x}) \, dy = \int \int_{U} 1 \, dxdy = A.
\]
(c)
\[
2\pi - 2A = \int_{-\pi}^{\pi} (1 - 2(x(s) - \overline{x})y'(s)) \, ds
= \int_{-\pi}^{\pi} ((x'(s))^2 + (y'(s))^2 - 2(x(s) - \overline{x})y'(s)) \, ds
= \int_{-\pi}^{\pi} ((x'(s))^2 - (x(s) - \overline{x})^2) \, ds + \int_{-\pi}^{\pi} ((x(s) - \overline{x}) + y'(s))^2 \, ds.
\]
The second integral in the last line above is clearly non-negative. And, by (a), the first integral is also non-negative. Thus, \( A \leq \pi \).
If $A = \pi$, then both integrals must be zero. By (a), the vanishing of the first integral implies that $x(s) = a + b \cos s + c \sin s$ for some constants $a, b$ and $c$. It’s clear that $a = \pi$. Then, the vanishing of the second integral implies $y'(s) = x(s) - \pi = b \cos s + c \sin s$, and, hence, $y(s) = d + b \sin s - c \cos s$ for some constant $d$. So $r(s) = (a + b \cos s + c \sin s, d + b \sin s - c \cos s), -\pi \leq s \leq \pi$, which is a parametrization of the circle $(x - a)^2 + (y - d)^2 = b^2 + c^2$. Since the length is $r$ is $2\pi$, the radius is 1.