HOMEWORK 3 SOLUTIONS

1. 3.3.18

Solution. When \( r \neq 0 \), we have

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r^2} + \frac{1}{r^2} + 0 = 0.
\]

2. Let \( G \) be an open region in \( \mathbb{R}^2 \), \((x_0, y_0)\) a point in \( G \), and \( u(x, y) \) a harmonic function defined on \( G \). Denote by \( C_r \) the circle centered at \((x_0, y_0)\) with radius \( r \).

(a) Suppose \( r > s > 0 \), and the closed disc bounded by \( C_r \) is contained in \( G \). Let \( v(x, y) = \ln \sqrt{(x-x_0)^2 + (y-y_0)^2} \). Show that \( \int_{C_s} u \frac{\partial v}{\partial n} \, ds = \int_{C_r} u \frac{\partial v}{\partial n} \, ds \), where \( \vec{n} \) is the outward unit normal vector of these circles.

(b) Show that, if the closed disc bounded by \( C_r \) is contained in \( G \), then \( u(x_0, y_0) = \frac{1}{2\pi r} \int_{C_r} u(x, y) \, ds \).

(c) Assume \( G \) is connected. Show that \( u(x, y) \) can not attain its maximum or minimum in \( G \) unless it’s a constant.

(d) Suppose that \( G \) is bounded, and has smooth boundary \( \partial G \). Let \( u_0 \) be a continuous function defined on \( \partial G \). Show that the solution of the Dirichlet problem

\[
\begin{cases}
\Delta u = 0 & \text{in } G, \\
u = u_0 & \text{on } \partial G,
\end{cases}
\]

if exists, is unique.

Proof. (a) From the previous problem, we know that \( v \) is harmonic except at \((x_0, y_0)\). Let \((\rho, \theta)\) be the polar coordinates centered at \((x_0, y_0)\). Then

\[
\nabla v = \rho^{-2}(x-x_0, y-y_0) = \frac{\vec{n}}{\rho},
\]

and

\[
\frac{\partial v}{\partial n} = \frac{\partial v}{\partial \rho} = \frac{1}{\rho}.
\]
Denote by $A$ the annulus between $C_s$ and $C_r$, and by $D_\rho$ the disc bounded by $C_\rho$. By Green’s formula, we have
\[
\int_{C_r} u \frac{\partial v}{\partial n} \, ds - \int_{C_s} u \frac{\partial v}{\partial n} \, ds = \int \int_A (u \Delta v + \nabla v \cdot \nabla u) \, dxdy
\]
\[
= \int \int_A \nabla v \cdot \nabla u \, dxdy
\]
\[
= \int_r^r \int_0^{2\pi} \nabla u \cdot \frac{n}{\rho} \rho \, d\theta d\rho
\]
\[
= \int_r^r \int_0^{2\pi} \nabla u \cdot \frac{n}{\rho} \, d\theta d\rho
\]
\[
= \int_r^r \frac{1}{\rho} \left( \int_{C_\rho} \frac{\partial v}{\partial n} \, ds \right) \, d\rho
\]
\[
= \int_r^r \frac{1}{\rho} \left( \int \int_{D_\rho} \Delta u \, dxdy \right) \, d\rho
\]
\[
= 0.
\]

(b) From (a), we know that
\[
\int_{C_r} u \frac{\partial v}{\partial n} \, ds = \lim_{s \to 0} \int_{C_s} u \frac{\partial v}{\partial n} \, ds = \lim_{s \to 0} \int_{C_s} u \cdot \frac{1}{\rho} \, ds
\]
\[
= \lim_{s \to 0} \int_0^{2\pi} u(x_0 + s \cos \theta, y_0 + s \sin \theta) \, d\theta = 2\pi \cdot u(x_0, y_0).
\]
But $\frac{\partial v}{\partial n} = \frac{1}{r}$ on $C_r$. Thus,
\[
u(x_0, y_0) = \frac{1}{2\pi} \int_{C_r} u \frac{\partial v}{\partial n} \, ds = \frac{1}{2\pi r} \int_{C_r} u(x, y) \, ds.
\]

(c) Say $u$ attains its maximum at $(x_0, y_0)$ in $G$. From (b), we know that the average of $u$ on any "small" circle centered at $(x_0, y_0)$ equals $u(x_0, y_0)$. So $u$ must also attain its maximum at very point on these circles. That is to say $u$ attains it maximum at every point in a "small" neighborhood of $(x_0, y_0)$. We can then begin with any other point in this neighborhood and repeat the same argument to show that $u$ actually attains its maximum on a larger set. Repeat this process. Since $G$ is connected, we will eventually get that $u$ attains its maximum at every point in $G$, i.e., $u$ is a constant. This proves (c). (Sorry, I cheated a little bit here. A precise proof requires some knowledge of point set topology.)

(d) Without loss of generality, we assume that $G$ is connected. (Otherwise, consider each of $G$’s connected components.) Let $u$, $v$ be two solutions to this Dirichlet problem, and $w = u - v$. Then $w$ is a solution to the following Dirichlet problem:
\[
\begin{align*}
\triangle w &= 0 \quad \text{in } G, \\
w &= 0 \quad \text{on } \partial G.
\end{align*}
\]
By (c), $w$ can not attain its maximum or minimum in the interior. So it must attain both its maximum and minimum on the boundary. But $w = 0$ on the boundary. This
implies \( \max w = \min w = 0 \), i.e., \( w = 0 \) everywhere. Thus \( u = v \) everywhere. This proves the uniqueness of the solution.