HOMEWORK 3

1. 3.3.18

2. Let $G$ be an open region in $\mathbb{R}^2$, $(x_0, y_0)$ a point in $G$, and $u(x, y)$ a harmonic function defined on $G$. Denote by $C_r$ the circle centered at $(x_0, y_0)$ with radius $r$.

   (a) Suppose $r > s > 0$, and the closed disc bounded by $C_r$ is contained in $G$. Let $v(x, y) = \ln \sqrt{(x-x_0)^2 + (y-y_0)^2}$. Show that $\int_{C_s} u \frac{\partial v}{\partial n} ds = \int_{C_r} u \frac{\partial v}{\partial n} ds$, where $\mathbf{n}$ is the outward unit normal vector of these circles.

   (b) Show that, if the closed disc bounded by $C_r$ is contained in $G$, then $u(x_0, y_0) = \frac{1}{2\pi r} \int_{C_r} u(x, y) ds$.

   (c) Assume $G$ is connected. Show that $u(x, y)$ can not attain its maximum or minimum in $G$ unless it’s a constant.

   (d) Suppose that $G$ is bounded, and has smooth boundary $\partial G$. Let $u_0$ be a continuous function defined on $\partial G$. Show that the solution of the Dirichlet problem

   $$\begin{cases}
   \Delta u = 0 & \text{in } G, \\
   u = u_0 & \text{on } \partial G,
   \end{cases}$$

   if exists, is unique.