Problem Set 1 [Revised]

Due: 23 February 2012 in class

1. (10 points) Show that

\[
L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & 0 & 1 \end{pmatrix}
\]

is the inverse of

\[
S = \begin{pmatrix} 1 & 0 & 0 \\ -\ell_{21} & 1 & 0 \\ -\ell_{31} & 0 & 1 \end{pmatrix}
\]

2. (10 points) By trial and error, find examples of 2 by 2 matrices such that

(a) \(LU \neq UL\)

(b) \(A^2 = -I\), with only real entries in \(A\)

(c) \(B^2 = 0\), with no zeros in \(B\)

3. (10 points) Factor the matrix \(A = LU\), where \(L\) is lower triangular, \(U\) is upper triangular, and

\[
A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}
\]

4. (10 points) Use back substitution twice to solve \(LUx = f\), where

\[
L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 7 \end{pmatrix}, \quad \text{and} \quad f = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}
\]

5. (20 points) Consider a line of \(n\) nodes, each connected to its neighbors by a resistor of resistance \(R\). At the first node, potential is set to 1. At the \(n\)th node, potential is set to 0.

(a) Write down \(n\) equations relating \(v_1, v_2, \ldots, v_n\). For \(n = 5\), write out by hand the equations in the form \(Ax = b\).

(b) Write a Matlab program that, for arbitrary \(n\), forms \(A\) and \(b\) and solves for \(x\). At approximately what value of \(n\) does it take 1 second to solve the linear system. Make sure you use the sparse data structure for \(A\)

6. (20 points) Consider the 2d lattice of points from \((1, 1)\) to \((n, n)\). Each is connected to its neighbors by a resistor of resistance \(R\). At the first node \(v_1 = 1\). At the last node, \(v_{n^2} = 0\).
(a) In the $n = 3$ case, write out by hand the 9 linear equations in the form $Ax = b$.

(b) Write a Matlab program that, for arbitrary $n$, forms $A$ and $b$ and solves for $x$. At approximately what value of $n$ does it take 1 second to solve the linear system. Make sure you use the sparse data structure for $A$.

(c) Based on the results of 5b and 6b: is a one-dimensional problem involving a million nodes more, less, or equally expensive as a two-dimensional problem involving a million nodes?