**SL₂(k)-modules**

Let \( L = sl_2(k) \). A basis is given by:

\[
    e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
    f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
    g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

We have \([e, f] = h, [h, e] = 2e, [h, f] = -2f\). Thus, \( h \) is semisimple. Since \( L \) is simple, it is semisimple. Let \( V \) be an \( L \)-module, \( \dim V < \infty \). Then \( h : V \to V \) is semisimple. Thus \( V = \oplus_{\lambda \in k} V_{\lambda} \) where \( V_{\lambda} = \{ v \in V | hv = \lambda v \} \).

If \( v \in V_{\lambda} \) then \( ev \in V_{\lambda + 2}, fv \in V_{\lambda - 2} \).

Assume now that \( V \) is irreducible. We can find \( v_0 \in V - \{0\} \) such that \( ev_0 = 0 \). Set \( v_{-1} = 0, v_n = \frac{1}{n!}v_0, n \in \mathbb{N} \). We have

(a) \( hv_n = (\lambda - 2n)v_n \) for \( n \geq -1 \)

(b) \( fv_n = (n + 1)v_{n+1} \) for \( n \geq -1 \)

(c) \( ev_n = (\lambda - n + 1)v_{n-1} \) for \( n \geq 0 \).

(c) is shown by induction on \( n \). For \( n = 0 \) it is clear. Assuming \( n \geq 1 \),

\[
    ev_n = n^{-1}efv_{n-1} = n^{-1}hv_{n-1} + n^{-1}fev_{n-1} \\
    = n^{-1}(\lambda - 2n + 2)v_{n-1} + n^{-1}f(\lambda - n + 2)v_{n-2} \\
    = n^{-1}(\lambda - 2n + 2)v_{n-1} + n^{-1}(\lambda - n + 2)(n - 1)v_{n-1} = (\lambda - n + 1)v_{n-1}.
\]

By (a), the non-zero \( v_n \) are linearly independent. Since \( \dim V < \infty \), there exists \( m \geq 0 \) such that \( v_0, v_1, \ldots, v_m \) are \( \neq 0 \) and \( v_{n+1} = 0 \). Then \( v_{m+1} = v_{m+2} = \cdots = 0 \). Now \( v_0, v_1, \ldots, v_m \) form a basis of an \( L \)-submodule which must be the whole of \( V \). Now (c) with \( n = m + 1 \) gives \( 0 = (\lambda - m)v_n \) hence \( \lambda = n \). Thus the action of \( e, f, h \) in the basis \( v_0, v_1, \ldots, v_m \) is

\[
    hv_n = (m - 2n)v_n \text{ for } n \in [0, m] \\
    fv_n = (n + 1)v_{n+1} \text{ for } n \in [0, m] \\
    ev_n = (m - n + 1)v_{n-1} \text{ for } n \in [0, m]
\]

with the convention \( v_{-1} = 0, v_{m+1} = 0 \).

Conversely, given \( m \geq 0 \) we can define an \( L \)-module structure on an \( m + 1 \) dimensional vector space with basis \( v_0, v_1, \ldots, v_m \) by the formulas above. Thus we have a 1-1 correspondence between the set of isomorphism classes of irreducible \( L \)-modules and the set \( \mathbb{N} \).

Now let \( V \) be any finite dimensional \( L \)-module. Then:
(a) the eigenvalues of \( h : V \to V \) are integers; the multiplicity of the eigenvalue \( a \) equals that of \(-a\).

(b) If \( h : V \to V \) has an eigenvalue in \( 2\mathbb{Z} \) then it has an eigenvalue \( 0 \).

(c) If \( h : V \to V \) has an eigenvalue in \( 2\mathbb{Z} + 1 \) then it has an eigenvalue \( 1 \).

Indeed, by Weyl, we are reduced to the case where \( V \) is irreducible; in that case we use the explicit description of \( L \) given above.

**ROOTS**

Let \( L \) be a semisimple Lie algebra \( \neq 0 \). A subalgebra \( T \) of \( L \) is said to be **toral** if any element of \( T \) is semisimple in \( L \).

**Lemma.** If \( T \) is toral then \( T \) is abelian.

Let \( x \in T \). Assume that \( \text{ad}(x) : T \to T \) has some eigenvalue \( a \neq 0 \). Thus \([x, y] = ay\) for some \( y \in T - \{0\} \). Now \( \text{ad}(y) : L \to L \) is semisimple hence \( \text{ad}(y) : T \to T \) is semisimple hence \( x = \sum_j u_j \) where \( u_j \in T \) are eigenvectors of \( \text{ad}(y) : T \to T \) with corresponding eigenvalue \( \lambda_j \). Hence \( \text{ad}(y)x = \sum_{j; \lambda_j \neq 0} \lambda_j u_j \).

But \( \text{ad}(y)x = -ay \). But \( y \) is in the 0-eigenspace of \( \text{ad}(y) \) and \( \sum_{j; \lambda_j \neq 0} \lambda_j u_j = -ay \) is a contradiction. Thus, all eigenvalues of \( \text{ad}(x) : T \to T \) are 0. Now \( \text{ad}(x) : L \to L \) is semisimple hence \( \text{ad}(x) : T \to T \) is semisimple hence \( \text{ad}(x) : T \to T \) is 0. The lemma follows.

Let \( H \) be a maximal toral subalgebra of \( L \). Now \( \{\text{ad}(h) : L \to L | h \in H\} \) is a family of commuting semisimple endomorphisms of \( L \). Hence \( L = \bigoplus \alpha L_{\alpha} \) where \( \alpha \) runs over the dual space \( H^* \) of \( H \) and \( L_{\alpha} = \{x \in L | [h, x] = \alpha(h)x \forall h \in H\} \). Now \( L_0 = \{x \in L | [h, x] = 0 \forall h \in H\} \) and \( H \subset L_0 \) by the lemma. We say that \( \alpha \in H^* \) is a root or \( \alpha \in R \) if \( \alpha \neq 0 \) and \( L_\alpha \neq 0 \). We have \( L = L_0 \oplus \bigoplus_{\alpha \in R} L_\alpha \) (root decomposition or Cartan decomposition).

**Lemma.** (a) For any \( \alpha, \beta \in H^* \) we have \( [L_{\alpha}, L_{\beta}] \subset L_{\alpha+\beta} \).

(b) If \( x \in L_{\alpha}, \alpha \neq 0 \) then \( \text{ad}(x) \) is nilpotent.

(c) If \( \alpha, \beta \in H^*, \alpha + \beta \neq 0 \) then \( \kappa(L_{\alpha}, L_{\beta}) = 0 \).

(a) Let \( x \in L_{\alpha}, y \in L_{\beta} \). For \( h \in H \) we have
\[
[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \alpha(h)[x, y] + \beta(h)[x, y] = (\alpha + \beta)(h)[x, y]
\]
hence
\[ [x, y] \in L_{\alpha+\beta} \).

(b) For any \( \beta \in H^* \) we have \( n\alpha + \beta \notin R \) for large \( n \) hence using (a), \( \text{ad}(x)^nL_\beta = 0 \). Now (b) follows.

(c) We can find \( h \in H \) with \( (\alpha + \beta)(h) \neq 0 \). Let \( x \in L_{\alpha}, y \in L_{\beta} \). We have \( \kappa([h, x], y) = \kappa([y, h], x) \) hence \( \alpha(h)\kappa(x, y) = -\beta(h)\kappa(x, y) \). Thus \( (\alpha + \beta)(h)\kappa(x, y) = 0 \) and \( \kappa(x, y) = 0 \).

**Lemma.** The restriction of \( \kappa \) to \( L_0 \) is non-singular.

**Proposition.** \( L_0 = H \).

We show:
(a) If \( x \in L_0 \) and \( x = s + n \) is a Jordan decomposition in \( L \) then \( s \in L_0, n \in L_0 \).
We have \( \text{ad}(x)H \subset \{0\} \) hence \( \text{ad}(x)_sH \subset \{0\}, \text{ad}(x)_nH \subset \{0\} \), hence \( \text{ad}(s)H \subset \{0\}, \text{ad}(n)H \subset \{0\} \), hence \( s \in L_0, n \in L_0 \).

(b) If \( x \in L_0 \) is semisimple in \( L \) then \( x \in H \).
From the assumption, \( H + kx \) is a toral algebra hence it is \( H \) by the maximality of \( H \). Hence \( x \in H \).

(c) The restriction of \( \kappa \) to \( H \) is non-singular.
Assume that \( h \in H \) and \( \kappa(h, H) = 0 \). Let \( x = s + n \in L_0 \) be as in (a).
Then \( s \in L_0, n \in L_0 \). By (a) we have \( s \in H \) hence \( \kappa(h, s) = 0 \). Now \( \text{ad}(n) : L \to L \) is nilpotent and \( \text{ad}(n), \text{ad}(h) \) commute hence \( \text{ad}(h)\text{ad}(n) \) is nilpotent hence \( \text{tr}(\text{ad}(h)\text{ad}(n), L) = 0 \). Thus \( \kappa(h, n) = 0 \). Hence \( \kappa(h, x) = 0 \). Thus \( \kappa(h, L_0) = 0 \).
Since \( \kappa|_{L_0} \) is non-singular, we have \( h = 0 \).

(d) \( L_0 \) is nilpotent.
By Engel it is enough to show that, if \( x \in L_0 \) then \( \text{ad}(x) : L_0 \to L_0 \) is nilpotent.
Write \( x = s + n \) as in (a). Now \( \text{ad}(s) : L_0 \to L_0 \) is 0 since \( s \in H \) (by (b)). Also \( \text{ad}(n) : L \to L \) is nilpotent hence \( \text{ad}(x) = \text{ad}(n) : L_0 \to L_0 \) is nilpotent.

(e) \( [L_0, L_0] \cap H = 0 \).
Let \( x \in [L_0, L_0] \cap H \). Write \( x = \sum_i [x_i, y_i] \) where \( x_i, y_i \in L_0 \). If \( h \in H \) we have \( \kappa(h, x) = \sum_i \kappa(h, [x_i, y_i]) = \sum_i \kappa(x_i, [y_i, h]) = 0 \) since \( [y_i, h] = 0 \). Thus \( \kappa(h, x) = 0 \). Since \( x \in H \) we see from (c) that \( x = 0 \).

(f) \( [L_0, L_0] = 0 \).
Otherwise, we have \( [L_0, L_0] \neq 0 \). Since \( L_0 \) is nilpotent and \( [L_0, L_0] \) is a non-zero ideal, we then have \( [L_0, L_0] \cap \text{centre}(L_0) \neq 0 \) (by a corollary of Engel). Let \( x \in [L_0, L_0] \cap \text{centre}(L_0), x \neq 0 \). Write \( x = s + n \) as in (a). Since \( \text{ad}(x)(L_0) \subset 0 \) we have \( \text{ad}(x)_n(L_0) \subset 0 \) hence \( \text{ad}(n)(L_0) \subset 0 \) hence \( n \in \text{centre}(L_0) \). Hence for any \( x' \in L_0, \text{ad}(x'), \text{ad}(n) : L \to L \) commute and \( \text{ad}(n) \) is nilpotent hence \( \text{ad}(x')\text{ad}(n) : L \to L \) is nilpotent hence \( \text{tr}(\text{ad}(x')\text{ad}(n), L) = 0 \) hence \( \kappa(x', n) = 0 \). Thus \( \kappa(L_0, n) = 0 \). Since \( \kappa|_{L_0} \) is non-singular we have \( n = 0 \). Thus \( x = s \in H \) (see (b)). Hence \( x \in [L_0, L_0] \cap H \) which is 0 by (e). Hence \( x = 0 \) a contradiction.

(g) If \( x \in L_0 \) is nilpotent then \( x = 0 \).
For all \( y \in L_0, \text{ad}(x), \text{ad}(y) \) commute and \( \text{ad}(x) \) is nilpotent hence \( \text{ad}(x)\text{ad}(y) : L \to L \) is nilpotent hence \( \text{tr}(\text{ad}(x)\text{ad}(y), L) = 0 \). Hence \( \kappa(x, y) = 0 \). Hence \( \kappa(x, L_0) = 0 \). Since \( \kappa|_{L_0} \) is non-singular we have \( x = 0 \).

We can now prove the proposition. Let \( x \in L_0 \). Write \( x = s + n \) as in (a). Then \( s \in L_0, n \in L_0 \). By (g) we have \( n = 0 \). By (b) we have \( s \in H \). Hence \( x \in H \). The proposition is proved.

Properties of roots.
Let \( \xi \in H^* \). Since \( \kappa|_H \) is non-singular there exists a unique element \( t_\xi \in H \) such that \( \xi(h) = \kappa(t_\xi, h) \) for all \( h \in H \). Now \( \xi \mapsto t_\xi \) is an isomorphism \( H^* \xrightarrow{\sim} H \).

(a) \( R \) spans the vector space \( H^* \).
If not, we can find \( h \in H, h \neq 0 \) so that \( \alpha(h) = 0 \) for all \( \alpha \in R \). Then \( [h, L_\alpha] = 0 \) for all \( \alpha \in R \). Also \( [h, L_0] = 0 \) since \( L_0 = H \) is abelian. Hence \( [h, L] = 0 \) so that \( h \in Z(L) \). But \( Z(L) = 0 \) since \( L \) is semisimple. Thus \( h = 0 \), contradiction.
(b) If \( \alpha \in R \) then \(-\alpha \in R \).

Assume that \(-\alpha \notin R \). Then \( L_{-\alpha} = 0 \). Hence \( \kappa(L_\alpha, L_\beta) = 0 \) for any \( \beta \in H^* \) hence \( \kappa(L_\alpha, L) = 0 \). Since \( \kappa \) is non-singular we have \( L_\alpha = 0 \), absurd.

(c) If \( \alpha \in R, x \in L_\alpha, y \in L_{-\alpha} \) then \( [x, y] = \kappa(x, y)t_\alpha \).

Let \( h \in H \). We have \( \kappa(h, [x, y]) = \kappa(y, [h, x]) = \alpha(h)\kappa(y, x) = \kappa(t_\alpha, h)\kappa(y, x) \) hence \( \kappa(h, [x, y] - \kappa(x, y)t_\alpha) = 0 \). Thus \( \kappa([x, y] - \kappa(x, y)t_\alpha, H) = 0 \). Since \( [x, y] - \kappa(x, y)t_\alpha \in H \) and \( \kappa_H \) is non-singular, we have \( [x, y] - \kappa(x, y)t_\alpha = 0 \).

(d) Let \( \alpha \in R \) and let \( x \in L_\alpha - \{0\} \neq 0 \). There exists \( y \in L_{-\alpha} \) such that \( \kappa(x, y) \neq 0 \).

Assume that \( \kappa(x, L_{-\alpha}) = 0 \). Then \( \kappa(x, L_\beta) = 0 \) for any \( \beta \in H^* \) hence \( \kappa(x, L) = 0 \) hence \( x = 0 \) absurd.

(e) Let \( \alpha \in R \). We have \( \alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0 \).

The equality comes from the definition of \( t_\alpha \). Assume that \( \alpha(t_\alpha) = 0 \). Then \( [t_\alpha, L_\alpha] = 0, [t_\alpha, L_{-\alpha}] = 0 \). Let \( x, y \) be as in (d). We can assume that \( \kappa(x, y) = 1 \). Then \( [x, y] = t_\alpha \). Let \( S = kx + ky + kt_\alpha \), a Lie subalgebra of \( L \). We have \( [S, S] = kt_\alpha, [kt_\alpha, kt_\alpha] = 0 \) hence \( S \) is solvable. By Lie’s theorem for \( ad : S \to End(L) \) we see that \( ad(x') : L \to L \) is nilpotent for any \( x' \in [S, S] \). In particular \( ad(t_\alpha) : L \to L \) is nilpotent. Since \( t_\alpha \in H \) and all elements of \( H \) are semisimple, we see that \( ad(t_\alpha) : L \to L \) is also semisimple hence is 0. Thus \( t_\alpha \in Z(L) = 0 \).

This contradicts \( t_\alpha \neq 0 \).

(f) Let \( \alpha \in R \). Let \( x \in L_\alpha, x \neq 0 \). We can find \( y \in L_{-\alpha} \) such that, setting \( h = [x, y] \in H \) we have \( [h, x] = 2x, [h, y] = -2y \).

By (d),(e) we can find \( y \in L_{-\alpha} \) such that \( \kappa(x, y) = 2/\alpha(t_\alpha) \). Then \( h = 2t_\alpha/\alpha(t_\alpha) \). Hence

\[
[h, x] = \left(2/\alpha(t_\alpha)\right)t_\alpha, x = \left(2/\alpha(t_\alpha)\right)\alpha(t_\alpha)x = 2x,
\]

\[
[h, y] = \left(2/\alpha(t_\alpha)\right)[t_\alpha, y] = \left(2/\alpha(t_\alpha)\right)(-\alpha(t_\alpha)y) = -2y.
\]

(g) Let \( \alpha \in R \). Let \( x, y, h \) be as in (f). Then \( S = kx + ky + kh \) is a Lie subalgebra of \( L \) and \( e \to x, f \to y, h \to h \) is an isomorphism of Lie algebras \( sl_2(k) \sim S \).

This is clear.

(h) Let \( \alpha \in R \). Let \( h_\alpha = 2t_\alpha/\alpha(t_\alpha) \) (see (f)). We have \( h_\alpha = -h_{-\alpha} \).

It suffices to show that \( t_\alpha = -t_{-\alpha} \). Since \( \kappa|H \) is non-singular it suffices to show that, for any \( h \in H \) we have \( \kappa(h, t_\alpha) = -\kappa(h, t_{-\alpha}) \) or that \( \alpha(h) = -(\alpha(h)) \). This is clear.

(i) Let \( \alpha \in R \). Then \( 2\alpha \notin R \).

Let \( x, y, h, S \) be as in (g). Let \( M = \bigoplus_{c \in \kappa L_\alpha} \) is an \( S \)-module under \( ad \) and \( h \) acts on \( L_\alpha \) as multiplication by \( c\alpha(h) = c\alpha(2t_\alpha)/\alpha(t_\alpha) = 2c \). By representation theory of \( sl_2(k) \) (see below) the eigenvalues of \( h : M \to M \) are integers. Hence \( M = \bigoplus_{c \in \kappa(1/2)Z} L_\alpha \). Now \( S + H \) is an \( S \)-submodule of \( M \). By Weyl, there exists an \( S \)-submodule \( M' \) of \( M \) such that \( M = (S + H) \oplus M' \). Now the 0-eigenspace of \( h : M \to M \) is \( L_0 = H \) hence it is contained in \( S + H \). Thus the 0-eigenspace of \( h : M' \to M' \) is 0. Hence \( h : M' \to M' \) does not have eigenvalues in \( 2\mathbb{Z} \). The eigenvalues of \( h : S + H \to S + H \) are 0, 2, -2. We see that 4 is not an eigenvalue of \( h : M \to M \).
If we had $2\alpha \in R$ then a non-zero-vector in $L_{2\alpha}$ would be an eigenvector of $h : M \to M$ with eigenvalue 4, contradiction.

(j) Let $\alpha \in R$. Then $\alpha/2 \notin R$.

If we had $\alpha/2 \in R$ then applying (i) to $\alpha/2$ we would deduce that $\alpha \notin R$, contradiction.

(k) In (i) we have $M' = 0$.

From (j) we see that $L_{\alpha/2} = 0$ hence the 1-eigenspace of $h : M \to M$ is 0. Thus $h : M' \to M'$ has no eigenvalue 1 (nor 0, see (i)). Hence $h : M' \to M'$ has no odd or even eigenvalues. Hence $M' = 0$.

(l) Let $\alpha \in R$. We have $\dim L_{\alpha} = 1$. Moreover $c\alpha \in R, c \in k$ implies $c \in \{1, -1\}$.

Let $x, y, h, S$ be as in (g). By (k) we have $\oplus_{c \in k} L_{c\alpha} = S + H$. The result follows.

(m) Let $\alpha, \beta \in R, \beta \neq \pm \alpha$. Let $h_\alpha = 2t_\alpha/\alpha(t_\alpha)$. Then $\beta(h_\alpha) \in Z$ and $\{n \in Z | \beta + n\alpha \in R\}$ is of the form $\{-r, -r + 1, \ldots, 0, \ldots, q - 1, q\}$ where $-r \leq 0 \leq q$.

Let $x, y, h, S$ be as in (g). Then $h = h_\alpha$. Let $K = \oplus_{n \in Z} L_{\beta + n\alpha} \subset L$. This is an $S = sl_2(k)$-module under $ad$ such that any eigenvalue of $h$ on $L_{\beta + n\alpha}$ is $\beta(h_\alpha) + 2n$. For $n = 0$ the eigenvalue is $\beta(h_\alpha)$ and it has multiplicity 0 hence $\beta(h_\alpha) \in Z$. We see that all eigenvalues have multiplicity one and they all have the same parity. It follows that the $S$-module $K$ is simple. (See below.) The result follows.

(n) Let $\alpha, \beta \in R, \beta \neq \pm \alpha$. Let $h_\alpha = 2t_\alpha/\alpha(t_\alpha)$. We have $\beta - \beta(h_\alpha)\alpha \in R$.

By (m), we have $\beta(h_\alpha) - 2r = -(\beta(h_\alpha) + 2q)$ that is $\beta(h_\alpha) = r - q$ and we must show that $-r \leq -\beta(h_\alpha) \leq q$ that is $-r \leq -r + q \leq q$. This is clear.

(o) If $\alpha, \beta, \alpha + \beta \in R$ then $[L_{\alpha}, L_{\beta}] = L_{\alpha + \beta}$.

Since $2\alpha \notin R$ we have $\beta \neq \pm \alpha$. Consider the irreducible $S = sl_2(k)$-module $K$ in (m). With notations in (m) we have $L_\beta = (r - q)$-eigenspace of $h : K \to K$ and $L_{\alpha + \beta} = (r - q + 2)$-eigenspace of $h : K \to K$. It is enough to show that $e \in sl_2(k)$ maps the $j$-eigenspace of $h : K \to K$ onto the $(j + 2)$-eigenspace (if both these eigenspaces are 1-dimensional). This follows from the explicit description of simple $sl_2(k)$-modules (see below).

(p) The smallest Lie subalgebra $L'$ of $L$ that contains $L_{\alpha}$ for all $\alpha \in R$ is $L$ itself.

It suffices to show that $L'$ contains $H$. From (a) it follows that $\{t_\alpha | \alpha \in R\}$ spans $H$ as a vector space. Hence it is enough to show that for $\alpha \in R$ we have $t_\alpha \in L'$. But by (c),(d) we have $t_\alpha \in [L_{\alpha}, L_{-\alpha}]$.

Rationality.

Define $(\cdot, \cdot) : H^* \times H^* \to k$ to be the symmetric bilinear form $(\xi, \xi') = \kappa(t_\xi, t_{\xi'}) = \sum_{\alpha \in R} \alpha(t_\xi)\alpha(t_{\xi'})$. This form is non-singular. For $\alpha \in R$ we have $(\xi, \alpha) = \kappa(t_\xi, t_{\alpha}) = \alpha(t_\xi)$. Hence $(\xi, \xi') = \sum_{\alpha \in R} (\xi, \alpha)(\xi', \alpha)$.

For $\alpha \in R$ we have $(\alpha, \alpha) = \kappa(t_\alpha, t_{\alpha}) \neq 0$. For $\alpha, \beta \in R$ we have $2(\alpha, \beta)/(\alpha, \alpha) = \kappa(2t_\alpha/\kappa(t_\alpha, t_{\alpha}), t_{\beta}) = \kappa(h_\alpha, t_{\beta}) = \beta(h_\alpha) \in Z$.

Now from $(\beta, \beta) = \sum_{\alpha \in R} (\beta, \alpha)^2$ we deduce $4(\beta, \beta)^{-1} = \sum_{\alpha \in R} (2(\beta, \alpha)/(\beta, \beta))^2 \in Z$. Thus $(\beta, \beta) \in Q$ hence $(\alpha, \beta) \in Q$ for any $\alpha, \beta \in R$. 
Let $E$ be the $\mathbb{Q}$-subspace of $H^*$ spanned by $R$. Let $\alpha_1, \ldots, \alpha_n$ be a $k$-basis of $H^*$ contained in $R$. We show that $\alpha_1, \ldots, \alpha_n$ is a $\mathbb{Q}$-basis of $E$. Let $\alpha \in R$. We have $\alpha = \sum_{i=1}^n c_i \alpha_i$ with $c_i \in k$. It suffices to show that $c_i \in \mathbb{Q}$ for all $i$. For any $j \in [1, n]$ we have 

$$2(\alpha, \alpha_j)/\langle \alpha_j, \alpha_j \rangle = \sum_{i=1}^n c_i 2(\alpha_i, \alpha_j)/\langle \alpha_j, \alpha_j \rangle.$$ 

This is a linear system of $n$ equations with $n$ unknowns $c_i$ with non-zero determinant and integer coefficients. Hence $c_i \in \mathbb{Q}$ for all $i$. Hence $E$ coincides with the $\mathbb{Q}$-subspace of $H^*$ spanned by $\alpha_1, \ldots, \alpha_n$.

Let $\xi \in E, \xi \neq 0$. We have $\langle \xi, \xi \rangle = \sum_{\alpha \in R} \langle \xi, \alpha \rangle^2$. This is a rational number $\geq 0$. If it 0 then $\langle \xi, \alpha \rangle = 0$ for all $\alpha \in R$ hence $\xi = 0$. Thus $(,)|_E$ has rational values and is positive definite.

We may summarize the properties of $R \subset E$ and $(,)|_E$ as follows:

$R$ spans $E$ as a $\mathbb{Q}$-vector space, $0 \notin R$. If $\alpha \in R$ then $-\alpha \in R$ but $c\alpha \notin R$ if $c \in \mathbb{Q} - \{1, -1\}$. If $\alpha, \beta \in R$ then $2(\beta, \alpha)/\langle \alpha, \alpha \rangle \in \mathbb{Z}$ and $\beta - 2(\beta, \alpha)/\langle \alpha, \alpha \rangle \alpha \in R$. 