In this section the ground field $k$ is arbitrary.

Let $L$ be a Lie algebra. A pair $(U, i)$ where $U$ is an associative algebra and $i$ is a Lie algebra homomorphism $L \rightarrow U$ is called a universal enveloping algebra of $L$ if the following holds: if $U'$ is any associative algebra and $i'$ is a Lie algebra homomorphism $L \rightarrow U'$ then there exists a unique algebra homomorphism $f : U \rightarrow U'$ such that $i' = fi$.

**Lemma.** (a) Let $(U, i), (\tilde{U}, \tilde{i})$ be universal enveloping algebras of $L$. Then there exists a unique algebra isomorphism $j : U \iso U'$ such that $i = \tilde{i}j$.

(b) $U$ is generated as an algebra by $i(L)$.

(c) Let $L_1, L_2$ be Lie algebras. Let $(U_1, i_1), (U_2, i_2)$ be universal enveloping algebras of $L_1, L_2$. Let $f : L_1 \rightarrow L_2$ be a Lie algebra homomorphism. Then there exists a unique algebra homomorphism $\tilde{f} : U_1 \rightarrow U_2$ such that $i_2f = \tilde{f}i_1$.

(d) Let $I$ be an ideal of $L$ and let $\bar{I}$ be the ideal of $U$ generated by $i(L)$. Then $i : L \rightarrow U$ induces a Lie algebra homomorphism $\bar{j} : L/\bar{I} \rightarrow U/\bar{I}$ and $(U/\bar{I}, \bar{j})$ is a universal enveloping algebra of $L/\bar{I}$.

(e) There is a unique algebra anti-automorphism $\pi : U \rightarrow U$ such that $\pi i = -i$. We have $\pi^2 = 1$.

(f) There is a unique algebra homomorphism $\delta : U \rightarrow U \otimes U$ such that $\delta (i(a)) = i(a) \otimes 1 + 1 \otimes i(a)$ for all $a \in L$.

(g) If $D : L \rightarrow L$ is a derivation then there is a unique derivation $D' : U \rightarrow U$ such that $iD = D'i$.

(a)-(f) are standard. We prove (g). Let $U_2$ be the algebra of $2 \times 2$ matrices with entries in $U$. Define a linear map $i' : L \rightarrow U_2$ by

$$a \mapsto i(a) \begin{pmatrix} i(D(a)) & 0 \\ 0 & i(a) \end{pmatrix}$$

This is a Lie algebra homomorphism:

$$i'([a, b]) = \begin{pmatrix} i(a)i(b) - i(b)i(a) & i(a)i(D(b)) + i(a)i(D(a)) - i(b)i(D(b)) - i(D(b))i(a) \\ 0 & i(a)i(b) - i(b)i(a) \end{pmatrix} = i'(a)i'(b) - i'(b)i'(a).$$
Hence there is an algebra homomorphism $j : U \to U_2$ such that $i' = ji$. We have $j(x) = \frac{xy}{0}$ for all $x \in U$ where $y$ is uniquely determined by $x$. Indeed this is true for $x \in i(L)$ and these generate $U$. We set $y = D'(x)$ where $D' : U \to U$. Then $D'$ is a derivation of $U$ such that $iD = Di$.

Construction of a universal enveloping algebra. Let $T$ be the tensor algebra of $L$. By definition, $T = T_0 \oplus T_1 \oplus T_2 \oplus \ldots$ where $T_0 = k1, T_1 = L$ and $T_i = L \otimes L \otimes \ldots L$ ($i$ times). The algebra structure is characterize by

$$(x_1 \otimes \ldots \otimes x_i)(y_1 \otimes \ldots \otimes y_j) = x_1 \otimes \ldots \otimes x_i \otimes y_1 \otimes \ldots \otimes y_j.$$

Let $K$ be the ideal of $T$ generated by the elements of form $[a, b] = a \otimes b - b \otimes a$ with $a, b \in L$. Let $U = T/I$. Let $i : L \to U$ be the composition of the canonical maps $L \to T \to U$. We have

$i[a, b] - i(a)i(b) + i(b)i(a) = K - \text{coset of } [a, b] - a \otimes b + b \otimes a = K$.

Hence $i : L \to U$ is a Lie algebra homomorphism.

**Proposition.** $(U, i)$ is a universal enveloping algebra of $L$.

Let $\{u_j | j \in J\}$ be a basis of the vector space $L$. The monomials $u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n}$ (where $j_1, j_2, \ldots, j_n \in J$) form a basis of $T_n$. We assume that $J$ is ordered. Define

$\text{index}(u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n}) = \sum_{i < k} \eta_{ik}$

where $\eta_{ik} = 0$ if $j_i \leq j_k$ and $\eta_{ik} = 1$ if $j_i > j_k$. We have $\text{index}(u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n}) = 0$ if and only if $j_1 \leq j_2 \leq \ldots j_n$. In this case the monomial is said to be standard. We regard 1 as a standard monomial. Assume now that $j_k > j_{k+1}$; then

$\text{index}(u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n})$

$= 1 + \text{index}(u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_{k+1}} \otimes u_{j_k} \otimes \ldots \otimes u_{j_n})$.

**Lemma 1.** Every element $x \in T$ is congruent modulo $K$ to a linear combination of standard monomials.

We may assume that $x$ is a monomial. We may assume that $x$ has degree $n > 0$ and index $p$ and that the result is true for monomials of degree $< n$ or for monomials of degree $n$ and index $< p$. Assume $x = u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n}$ is not standard and suppose $j_k > j_{k+1}$. We have

$u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n}$

$= u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_{k+1}} \otimes u_{j_k} \otimes \ldots \otimes u_{j_n}$

$+ u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes (u_{j_k} \otimes u_{j_{k+1}} - u_{j_{k+1}} \otimes u_{j_k}) \otimes \ldots \otimes u_{j_n}$

$= u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_{k+1}} \otimes u_{j_k} \otimes \ldots \otimes u_{j_n}$

$+ u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes [u_{j_k}, u_{j_{k+1}}] \otimes \ldots \otimes u_{j_n} \mod K$.

The result follows from the induction hypothesis.

We now introduce the vector space $P_n$ with basis $u_{i_1} u_{i_2} \ldots u_{i_n}$ indexed by the various $i_1 \leq i_2 \leq \ldots i_n$ in $J$. Let $P = P_0 \oplus P_1 \oplus P_2 \oplus \ldots$. 
Lemma 2. There exists a linear map $\sigma : T \rightarrow P$ such that

(a) $\sigma(u_{i_1} \otimes u_{i_2} \otimes \ldots \otimes u_{i_n}) = u_{i_1} u_{i_2} \ldots u_{i_n}$ if $i_1 \leq i_2 \leq \ldots \leq i_n$,

(b) $\sigma(u_j \otimes u_{j_2} \otimes \ldots \otimes u_{j_n} - u_j \otimes u_{j_2} \otimes \ldots \otimes u_{j_{k+1}} \otimes u_{j_k} \otimes \ldots \otimes u_{j_n}) = \sigma(u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes [u_{j_k}, u_{j_{k+1}}] \otimes \ldots \otimes u_{j_n})$

for any $j_1, j_2, \ldots, j_n \in J$ and any $k$.

Let $T_{n,j}$ be the subspace of $T_n$ spanned by the monomials of degree $n$ and index $\leq j$. Define $\sigma(1) = 1$. Assume that $\sigma$ is already defined on $T_0 \oplus T_1 \oplus \ldots \oplus T_{n-1}$ and it satisfies (a),(b) for monomials of degree $< n$. We extend $\sigma$ linearly to $T_0 \oplus T_1 \oplus \ldots \oplus T_{n-1} \oplus T_{n,0}$ by requiring that $\sigma(u_{i_1} \otimes u_{i_2} \otimes \ldots \otimes u_{i_n}) = u_{i_1} u_{i_2} \ldots u_{i_n}$ for a standard monomial of degree $n$. Now assume that $i \geq 1$ and that $\sigma$ has already been defined on $T_0 \oplus T_1 \oplus \ldots \oplus T_{n-1} \oplus T_{n,i-1}$ so that (a),(b) is satisfied for monomials of degree of degree $< n - 1$ or for monomials of degree $n$ and index $< i$. Now let $u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n}$ be of index $i$. Suppose that $j_k > j_{k+1}$. We set

(*) $\sigma(u_j \otimes u_{j_2} \otimes \ldots \otimes u_{j_n}) = \sigma(u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_{k+1}} \otimes u_{j_k} \otimes \ldots \otimes u_{j_n} + \sigma(u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes [u_{j_k}, u_{j_{k+1}}] \otimes \ldots \otimes u_{j_n})$.

This makes sense. We show that (*) is independent of the choice of the pair $j_k > j_{k+1}$. Assume that we have another pair $j_l > j_{l+1}$. There are two cases: (1) $l > k + 1$, (2) $l = k + 1$.

Case (1). We set $u_{j_k} = u, u_{j_{k+1}} = v, u_{j_l} = w, u_{j_{l+1}} = t$. By the first definition

$$\begin{align*}
\sigma(u \otimes v \otimes \ldots \otimes w \otimes t) & = \sigma(\ldots v \otimes u \otimes \ldots \otimes w \otimes t \ldots) + [u, v] \otimes \ldots \otimes [u, v] \otimes w \otimes t \ldots) \\
& = \sigma(\ldots v \otimes u \otimes \ldots \otimes t \otimes w \ldots + v \otimes u \otimes \ldots \otimes [w, t] \ldots) \\
& + [u, v] \otimes \ldots \otimes t \otimes w \ldots + [u, v] \otimes \ldots \otimes [w, t] \ldots).
\end{align*}$$

The second definition leads to the same expression.

Case (2). We set $u_{j_k} = u, u_{j_{k+1}} = v = u_{j_l}, u_{j_{l+1}} = w$. By the first definition

$$\begin{align*}
\sigma(u \otimes v \otimes w \ldots) & = \sigma(\ldots v \otimes u \otimes w \ldots + [u, v] \otimes w \ldots) \\
& = \sigma(\ldots v \otimes w \otimes u \ldots + v \otimes [u, w] \ldots + [u, v] \otimes w \ldots) \\
& = \sigma(\ldots w \otimes v \otimes u \ldots + [v, w] \otimes u \ldots + v \otimes [u, w] \ldots + [u, v] \otimes w \ldots).
\end{align*}$$

By the second definition

$$\begin{align*}
\sigma(u \otimes v \otimes w \ldots) & = \sigma(u \otimes w \otimes v \ldots + u \otimes [v, w] \ldots) \\
& = \sigma(\ldots w \otimes u \otimes v \ldots + [u, w] \otimes v \ldots + u \otimes [v, w] \ldots) \\
& = \sigma(\ldots w \otimes v \otimes u \ldots + w \otimes [u, v] \ldots + [u, w] \otimes v \ldots + u \otimes [v, w] \ldots).
\end{align*}$$

Thus we are reduced to proving

$$\begin{align*}
\sigma([v, w] \otimes u \ldots + [v, u, w] \ldots + [u, v] \otimes w \ldots) & = \sigma(\ldots w \otimes [u, v] \ldots + [u, w] \otimes v \ldots + u \otimes [v, w] \ldots) \\
& = [v, w] \otimes u \ldots + [v, u, w] \ldots + [u, v] \otimes w \ldots)
\end{align*}$$

or equivalently $\sigma([v, w], u] \ldots + [v, u, w] \ldots + [u, v, w] \ldots) = 0$.

which follows from $[[v, w], u] + [v, [u, w]] = [[u, v], w] = 0$. The lemma is proved.
**Theorem (Poincaré-Birkhoff-Witt).** The standard monomials form a basis of \( U = T/K \).

By lemma 1 the standard monomials span \( U \). Now \( K \) is spanned by elements of the form
\[
    u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_n} - u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_k} \otimes \ldots \otimes u_{j_n} - u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes [u_{j_k}, u_{j_{k+1}}] \otimes \ldots \otimes u_{j_n},
\]
hence \( \sigma(K) = 0 \) and \( \sigma \) induces a linear map \( U \to P \). This linear map takes the standard monomials to linearly independent elements of \( P \). Hence the standard monomials are linearly independent in \( U \).

**Corollary.** The map \( i : L \to U \) is injective.

**Free Lie algebra.** Let \( X \) be a set. The free Lie algebra generated by \( X \) is a pair \((F, i)\) where \( F \) is a Lie algebra and \( i : X \to F \) is a map such that, if \( i' : X \to F' \) is a map of \( X \) into a Lie algebra, there is a unique Lie algebra homomorphism \( j : F \to F' \) such that \( i' = ji \). We show the existence of \((F, i)\). Let \( V \) be the vector space with basis \( X \). Let \( T \) be the tensor algebra of \( V \). Let \( F \) be the Lie subalgebra of \( T \) generated by \( X \). Then \( i \) is the obvious imbedding \( X \subset F \). Let \( i' : X \to F' \) be a map into a Lie algebra. This extends to a linear map \( V \to F' \). Let \( h : F' \to U' \) be the enveloping algebra of \( F' \). The composition \( V \to F' \xrightarrow{h} U' \) extends to an algebra homomorphism \( T \to U' \) and this restricts to a Lie algebra homomorphism \( a : F \to U' \). Now \( a(X) \subset h(F') \). Since \( F \) is generated by \( X \) as a Lie algebra, and \( h(F') \) is a Lie subalgebra, we see that \( a(F) \subset h(F') \). Since \( h \) is injective (by the PBW theorem) there exists a unique homomorphism of Lie algebras \( j : F \to F' \) such that \( F \xrightarrow{a} U' \) is equal to \( F \xrightarrow{j} F' \xrightarrow{h} U' \). This shows that \((F, i)\) is the free Lie algebra generated by \( X \).

### \( \mathfrak{sl}_2(k) \)-modules

Let \( L = \mathfrak{sl}_2(k) \). A basis is given by
\[
e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

We have \([e, f] = h, [h, e] = 2e, [h, f] = -2f \). Thus, \( h \) is semisimple. Since \( L \) is simple, it is semisimple. Let \( V \) be an \( L \)-module, \( \dim V < \infty \). Then \( h : V \to V \) is semisimple. Thus \( V = \bigoplus_{\lambda \in k} V_{\lambda} \) where \( V_{\lambda} = \{v \in V | hv = \lambda v\} \).

If \( v \in V_{\lambda} \) then \( ev \in V_{\lambda+2}, fv \in V_{\lambda-2} \).

Assume now that \( V \) is irreducible. We can find \( v_0 \in V - \{0\} \) such that \( v_0 \in V_{\lambda}, ev_0 = 0 \). Set \( v_{-1} = 0, v_n = \frac{f^n}{n!} v_0, n \in \mathbb{N} \). We have
(a) \( hv_n = (\lambda - 2n)v_n \) for \( n \geq -1 \)
(b) \( fv_n = (n + 1)v_{n+1} \) for \( n \geq -1 \)
(c) $e v_n = (\lambda - n + 1)v_{n-1}$ for $n \geq 0$.

(c) is shown by induction on $n$. For $n = 0$ it is clear. Assuming $n \geq 1$,

\[
e v_n = n^{-1}e f v_{n-1} = n^{-1}h v_{n-1} + n^{-1}f e v_{n-1}
= n^{-1}(\lambda - 2n + 2)v_{n-1} + n^{-1}f(\lambda - n + 2)v_{n-2}
= n^{-1}(\lambda - 2n + 2)v_{n-1} + n^{-1}(\lambda - n + 2)(n-1)v_{n-1} = (\lambda - n + 1)v_{n-1}.
\]

By (a), the non-zero $v_n$ are linearly independent. Since $\dim V < \infty$, there exists $m \geq 0$ such that $v_0, v_1, \ldots, v_m$ are $\neq 0$ and $v_{m+1} = 0$. Then $v_{m+2} = v_{m+3} = \cdots = 0$. Now $v_0, v_1, \ldots, v_m$ form a basis of an $L$-submodule which must be the whole of $V$. Now (c) with $n = m + 1$ gives $0 = (\lambda - m)v_n$ hence $\lambda = n$. Thus the action of $e, f, h$ in the basis $v_0, v_1, \ldots, v_m$ is

\[
h v_n = (m - 2n)v_n \text{ for } n \in [0, m]
f v_n = (n + 1)v_{n+1} \text{ for } n \in [0, m]
e v_n = (m - n + 1)v_{n-1} \text{ for } n \in [0, m]
\]
with the convention $v_{-1} = 0, v_{m+1} = 0$.

Conversely, given $m \geq 0$ we can define an $L$-module structure on an $m + 1$ dimensional vector space with basis $v_0, v_1, \ldots, v_m$ by the formulas above. Thus we have a 1-1 correspondence between the set of isomorphism classes of irreducible $L$-modules and the set $\mathbb{N}$.

Now let $V$ be any finite dimensional $L$-module. Then:

(a) the eigenvalues of $h : V \to V$ are integers; the multiplicity of the eigenvalue $a$ equals that of $-a$.

(b) If $h : V \to V$ has an eigenvalue in $2\mathbb{Z}$ then it has an eigenvalue 0.

(c) If $h : V \to V$ has an eigenvalue in $2\mathbb{Z} + 1$ then it has an eigenvalue 1.

Indeed, by Weyl, we are reduced to the case where $V$ is irreducible; in that case we use the explicit description of $L$ given above.

**A property of $\mathfrak{sl}_2$-modules**

Let $V$ be a $\mathfrak{sl}_2$-module such that $e : V \to V, f : V \to V$ are locally nilpotent. Then $\exp(e) : V \to V, \exp(-f) : V \to V$ are well defined isomorphisms. Hence $\tau = \exp(e) \exp(-f) \exp(e) : V \to V$ is a well defined isomorphism. For any integer $n$ let $V_n = \{x \in V | hx = nx\}$. Assume that $V = \oplus_n V_n$

**Lemma.** $\tau(V_n) \subset V_{-n}$.

Step 1. Assume that $V$ has a basis $\xi, \eta$ where $e \xi = 0, e \eta = \xi, f \xi = \eta, f \eta = 0, h \xi = \xi, h \eta = -\eta$.
We have $V = V_1 \oplus V_{-1}$ and $\exp(e) \xi = \xi, \exp(e) \eta = \eta = \xi, \exp(-f) \xi = \xi - \eta, \exp(-f) \eta = \eta$. It follows that $\tau(\xi) = -\eta, \tau(\eta) = \xi$. hence the result follows in this case.

Step 2. Assume that the result holds for $V$ and for $V'$. We show that it holds for $V \otimes V'$ where $x \in \mathfrak{sl}_2$ acts as $x \otimes 1 + 1 \otimes x$. 
A simple computation shows that for $x \in \mathfrak{sl}_2$, locally nilpotent, $\exp(x)$ acts on $V \otimes V'$ as $\exp(x) \otimes \exp(x)$. Hence $\tau$ acts on $V \otimes V'$ as $\tau \otimes \tau$. The result follows easily.

Step 3. If the result holds for $V$ then it holds for any direct summand of $V$ (as a $\mathfrak{sl}_2$-module).

(Obvious.)

Step 4. The result holds when $V$ is the irreducible module of dimension $n$.

(Induction on $n$.) This is obvious for $n = 1$ and is true for $n = 2$ by Step 1. Assume now that $n \geq 3$. Then $V$ is a direct summand of $V' \otimes V''$ where $V'$ is an irreducible module of dimension $n - 1$ and $V''$ is an irreducible module of dimension 2. By the induction hypothesis, the result holds for $V', V''$ hence it holds for $V' \otimes V''$ by Step 2 and for $V$ by Step 3.

Step 5. The result holds when $\dim V < \infty$.

Follows from the complete reducibility of $V$ and Step 4.

Step 6. The result holds in general.

Let $x \in V_n$. Let $N, N'$ be such that $e^{N+1}x = 0, f^{N'+1}x = 0$. The subspace of $V$ spanned by $f^i e^j x$ with $0 \leq j \leq N, 0 \leq i \leq N + N'$ is easily seen to be an $\mathfrak{sl}_2$-submodule $V'$. We have $\dim V' < \infty$. By Step 5 the result holds for $V'$. Hence $\tau(x) \in V_{-n}$. 