Problem set 5, due Thursday April 3

This homework is graded on 4 points; the first exercise is graded on 1 point, the second on 0.5 point, the third on 1 point, the fourth on 1.5 points. The final grade will be obtained by taking the minimum of 4 and the sum of the grades obtained in the 4 exercises.

(A) Formula for the stationary measure. Let $P$ be a transition probability on a finite state space $S$. Show that

1. $I - P$ has an eigenvalue equal to zero.
2. Assume that $I - P$ has only one eigenvalue equal to zero. Show that $P$ has a stationary measure and that it is given by

$$
\pi_i = \frac{\det((I - P)^{(i)})}{\sum_j \det((I - P)^{(j)})}
$$

where $M^{(j)}$ denotes the $(|S| - 1) \times (|S| - 1)$ obtained from $M$ by removing the $i$th row and column.
3. Let $P = (p_{i,j})_{1 \leq i,j \leq 3}$ be a transition probability on $\{1, 2, 3\}$ and set

$$
D(i, j, k) = p_{ji}(1 - p_{kk}) + p_{jk}p_{ki}
$$

Show that

(a) $\det((I - P)^{(1)}) = D(1, 2, 3)$, $\det((I - P)^{(2)}) = D(2, 3, 1)$, and $\det((I - P)^{(3)}) = D(3, 1, 2)$.
(b) Conclude that $P$ has a unique stationary distribution if and only if

$$
\Pi := D(1, 2, 3) + D(2, 3, 1) + D(3, 1, 2) > 0
$$

and then

$$
\pi_1 = \frac{D(1, 2, 3)}{\Pi}, \quad \pi_2 = \frac{D(2, 3, 1)}{\Pi}, \quad \pi_3 = \frac{D(3, 1, 2)}{\Pi}.
$$

(B) Balls in boxes. Consider two boxes 1 and 2 containing a total of $N$ balls. After the passage of each unit of time one ball is chosen randomly and moved to the other box. Consider the Markov chain with state space $\{0, 1, 2, \ldots, N\}$ representing the number of balls in box 1.

1. What is the transition matrix of the Markov chain?
2. Determine periodicity, transience, recurrence of the Markov chain.

(C) Constructing Markov chains with given stationary measure. Let $P$ be a transition probability on $S$ finite. Let $\pi$ be a probability vector with positive entries. In this exercise we construct two Markov chains with stationary measure $\pi$.

1. Metropolis chain.
   (a) Show that

$$
\hat{P}_{i,j} = \begin{cases} 
\frac{\pi_j P_{i,j}}{\pi_i P_{i,j}}, & \text{if } j \neq i \\
1 - \sum_{k \neq i} \frac{\pi_k P_{i,k}}{\pi_i P_{i,k}} & \end{cases}
$$

is a transition probability matrix so that $\pi \hat{P} = \pi$.
(b) Assume $P_{i,j} = P_{j,i}$ is aperiodic and irreducible. Show that $\hat{P}_{i,j}^n$ converges towards $\pi_j$ for any $j \in S$. 

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(2) Glauber dynamics. Set $S = S^V$ for two finite sets $S, V$ and put for $v \in V$

$$\Omega(x, v) = \{ y \in S^V : y_w = x_w \text{ for all } w \neq v \}$$

Put

$$\pi_{x,v}(y) = \begin{cases} \frac{\pi(y)}{\pi(\Omega(x,v))} & \text{if } y \in \Omega(x,v), \\ 0 & \text{if } y \notin \Omega(x,v) \end{cases}$$

Let $\tilde{P}$ be the Markov chain given by choosing independently at any step a vertex $v$ with the uniform measure on $V$ and choose a new configuration according to $\pi_{x,v}$.

(a) Describe the transition matrix $\tilde{P}$ and show that $\tilde{P}$ is a transition probability. Prove that $\pi$ is stationary for $\tilde{P}$.

(b) Show that it satisfies the weak Doeblin condition and determine recurrence/transience of the Markov chain.

(D) Particles in a region Consider a region $D$ of space containing $N$ particles. After the passage of each unit of time, each particle has probability $q \in (0, 1)$ of leaving region $D$, and $k$ new particles enter the region $D$ following a Poisson distribution with parameter $\lambda$:

$$P(x = k) = \frac{1}{k!} \lambda^k e^{-\lambda}.$$ The exit and entrance phenomena are assumed to be independent. Consider the Markov chain with state space $\mathbb{Z}_+ = \{0, 1, 2, \cdots \}$ representing the number of particles in the region.

(1) Compute the transition matrix $P$ for the Markov chain.

(2) Show that

$$\pi_k = e^{-\lambda} \frac{\lambda^k}{q^k k!}.$$ is stationary for $P$.

(3) Show that the Markov chain is irreducible.

(4) Show that there exists $B > 0$ such that $[0, B]$ is positive recurrent. Hint: Exhibit a Lyapunov function and use exercise D in Problem set 4.

(5) Show that for all $j$

$$\lim_{n \to \infty} P^n_{ij} = \pi_j$$

Hint: Show that $\|P(j, \cdot) - P(i, \cdot)\|_v \leq q|i - j|$ either by direct computation or by coupling techniques. Here $\|\mu\|_v = \frac{1}{2} \sum |\mu_i|$ is the total variation norm. Hint: Construct a coupling $X^i_1, X^i_1$ of $P(j, \cdot)$ and $P(i, \cdot)$ so that

$$\mathbb{E}[|X^i_1 - X^j_1|] \leq (1 - q)|j - i|.$$