MATRIX FACTORIZATIONS

1. \( A = LU = \begin{pmatrix} \text{lower triangular } L \\ 1's \text{ on the diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangular } U \\ \text{pivots on the diagonal} \end{pmatrix} \)

Requirements: No row exchanges as Gaussian elimination reduces square \( A \) to \( U \).

2. \( A = LDU = \begin{pmatrix} \text{lower triangular } L \\ 1's \text{ on the diagonal} \end{pmatrix} \begin{pmatrix} \text{pivot matrix } D \\ \text{is diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangular } U \\ 1's \text{ on the diagonal} \end{pmatrix} \)

Requirements: No row exchanges. The pivots in \( D \) are divided out to leave 1's on the diagonal of \( U \). If \( A \) is symmetric then \( U = L^T \) and \( A = LDL^T \).

3. \( PA = LU \) (permutation matrix \( P \) to avoid zeros in the pivot positions).

Requirements: \( A \) is invertible. Then \( P, L, U \) are invertible. \( P \) does all of the row exchanges on \( A \) in advance, to allow normal \( LU \). Alternative: \( A = L_1 P_1 U_1 \).

4. \( EA = R \) (\( m \) by \( m \) invertible \( E \)) (any \( m \) by \( n \) matrix \( A \)) = \text{rref}(A).

Requirements: None! The reduced row echelon form \( R \) has \( r \) pivot rows and pivot columns, containing the identity matrix. The last \( m - r \) rows of \( E \) are a basis for the left nullspace of \( A \); they multiply \( A \) to give \( m - r \) zero rows in \( R \). The first \( r \) columns of \( E^{-1} \) are a basis for the column space of \( A \).

5. \( S = C^T C = \begin{pmatrix} \text{lower triangular} \\ \text{upper triangular} \end{pmatrix} \) with \( \sqrt{D} \) on both diagonals

Requirements: \( S \) is symmetric and positive definite (all \( n \) pivots in \( D \) are positive). This Cholesky factorization \( C = \text{chol}(S) \) has \( C^T = L \sqrt{D} \), so \( S = C^T C = LDL^T \).

6. \( A = QR = \) (orthonormal columns in \( Q \)) (upper triangular \( R \)).

Requirements: \( A \) has independent columns. Those are orthogonalized in \( Q \) by the Gram-Schmidt or Householder process. If \( A \) is square then \( Q^{-1} = Q^T \).

7. \( A = X \Lambda X^{-1} = \) (eigenvectors in \( X \)) (eigenvalues in \( \Lambda \)) (left eigenvectors in \( X^{-1} \)).

Requirements: \( A \) must have \( n \) linearly independent eigenvectors.

8. \( S = Q \Lambda Q^T = \) (orthogonal matrix \( Q \)) (real eigenvalue matrix \( \Lambda \)) (\( Q^T \) is \( Q^{-1} \)).

Requirements: \( S \) is real and symmetric: \( S^T = S \). This is the Spectral Theorem.
9. \( A = BJB^{-1} = (\text{generalized eigenvectors in } B) \) (Jordan blocks in \( J \)) (\( B^{-1} \)).

**Requirements:** \( A \) is any square matrix. This Jordan form \( J \) has a block for each independent eigenvector of \( A \). Every block has only one eigenvalue.

10. \( A = U\Sigma V^T = \begin{pmatrix} \text{orthogonal} & \text{orthogonal} \\ U \text{ is } m \times m & V \text{ is } n \times n \end{pmatrix} \begin{pmatrix} \text{m} \times \text{n singular value matrix} \\ \sigma_1, \ldots, \sigma_r \text{ on its diagonal} \end{pmatrix} \begin{pmatrix} \text{orthogonal} \\ \text{1/\( \sigma_1 \), \ldots, 1/\( \sigma_r \) on diagonal} \end{pmatrix} \).

**Requirements:** None. This Singular Value Decomposition (SVD) has the eigenvectors of \( AA^T \) in \( U \) and eigenvectors of \( A^T A \) in \( V \); \( \sigma_i = \sqrt{\lambda_i(AA^T)} = \sqrt{\lambda_i(A^TA)} \).

11. \( A^+ = V\Sigma^+U^T = \begin{pmatrix} \text{orthogonal} & \text{orthogonal} \\ n \times n & m \times m \end{pmatrix} \begin{pmatrix} \text{n} \times \text{m pseudoinverse of } \Sigma \\ 1/\sigma_1, \ldots, 1/\sigma_r \text{ on diagonal} \end{pmatrix} \).  

**Requirements:** None. The pseudoinverse \( A^+ \) has \( A^+A = \text{projection onto row space of } A \) and \( AA^+ = \text{projection onto column space} \). \( A^+ = A^{-1} \) if \( A \) is invertible. The shortest least-squares solution to \( Ax = b \) is \( x^+ = A^+b \). This solves \( A^TAx^+ = A^Tb \).

12. \( A = QS = (\text{orthogonal matrix } Q) \) (symmetric positive definite matrix \( S \)).

**Requirements:** \( A \) is invertible. This polar decomposition has \( S^2 = A^TA \). The factor \( S \) is semidefinite if \( A \) is singular. The reverse polar decomposition \( A = KQ \) has \( K^2 = AA^T \). Both have \( Q = UV^T \) from the SVD.

13. \( A = UAU^{-1} = (\text{unitary } U) \) (eigenvalue matrix \( \Lambda \)) (\( U^{-1} \) which is \( U^H = U^T \)).

**Requirements:** \( A \) is normal: \( A^HA = AA^H \). Its orthonormal (and possibly complex) eigenvectors are the columns of \( U \). Complex \( \lambda \)'s unless \( S = S^H \): Hermitian case.

14. \( A = QTQ^{-1} = (\text{unitary } Q) \) (triangular \( T \) with \( \lambda \)'s on diagonal) \( (Q^{-1} = Q^H) \).

**Requirements:** Schur triangularization of any square \( A \). There is a matrix \( Q \) with orthonormal columns that makes \( Q^{-1}AQ \) triangular: Section 6.4.

15. \( F_n = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{n/2} \\ F_{n/2} \end{bmatrix} \sqrt{\text{even-odd permutation}} = \text{one step of the recursive FFT} \).

**Requirements:** \( F_n \) = Fourier matrix with entries \( w^{jk} \) where \( w^n = 1 \): \( F_nF_n^T = nI \). \( D \) has \( 1, w, \ldots, w^{n/2-1} \) on its diagonal. For \( n = 2^\ell \) the Fast Fourier Transform will compute \( F_nx \) with only \( \frac{1}{2} n\ell = \frac{1}{2} n \log_2 n \) multiplications from \( \ell \) stages of \( D \)’s.