Six Great Theorems of Linear Algebra

**Dimension Theorem** All bases for a vector space have the same number of vectors.

**Counting Theorem** Dimension of column space + dimension of nullspace = number of columns.

**Rank Theorem** Dimension of column space = dimension of row space. This is the rank.

**Fundamental Theorem** The row space and nullspace of \( A \) are orthogonal complements in \( \mathbb{R}^n \).

**SVD** There are orthonormal bases (\( v \)'s and \( u \)'s for the row and column spaces) so that \( Av_i = \sigma_i u_i \).

**Spectral Theorem** If \( A^T = A \) there are orthonormal \( q \)'s so that \( Aq_i = \lambda_i q_i \) and \( A = Q \Lambda Q^T \).

**LINEAR ALGEBRA IN A NUTSHELL**

**Nonsingular**
- \( A \) is invertible
- The columns are independent
- The rows are independent
- The determinant is not zero
- \( Ax = 0 \) has one solution \( x = 0 \)
- \( Ax = b \) has one solution \( x = A^{-1}b \)
- \( A \) has \( n \) (nonzero) pivots
- \( A \) has full rank \( r = n \)
- The reduced row echelon form is \( R = I \)
- The column space is all of \( \mathbb{R}^n \)
- The row space is all of \( \mathbb{R}^n \)
- All eigenvalues are nonzero
- \( A^T A \) is symmetric positive definite
- \( A \) has \( n \) (positive) singular values

**Singular**
- \( A \) is not invertible
- The columns are dependent
- The rows are dependent
- The determinant is zero
- \( Ax = 0 \) has infinitely many solutions
- \( Ax = b \) has no solution or infinitely many
- \( A \) has \( r < n \) pivots
- \( A \) has rank \( r < n \)
- \( R \) has at least one zero row
- The column space has dimension \( r < n \)
- The row space has dimension \( r < n \)
- Zero is an eigenvalue of \( A \)
- \( A^T A \) is only semidefinite
- \( A \) has \( r < n \) singular values