Problem Set 9.1, page 436

1. (a) The eigenvalues are $4, -2 + 2i, 2 \cos \theta$ and products $5, -2i, 1$. Note $(e^{i\theta})(e^{-i\theta}) = 1$.

2. In polar form these are $\sqrt{5}e^{i\theta}, 5e^{2i\theta}, \frac{1}{\sqrt{5}}e^{-i\theta}, \sqrt{5}$.

3. The absolute values are $r = 10, 100, \frac{1}{10}$, and 100. The angles are $\theta, 2\theta, -\theta$ and $-2\theta$.

4. $|z \times w| = 6, |z + w| \leq 5, |z/w| = \frac{2}{3}, |z - w| \leq 5$.

5. $a + ib = \frac{\sqrt{2}}{2} + \frac{1}{2} i, \frac{1}{2} + \frac{\sqrt{2}}{2} i, i, -\frac{1}{2} + \frac{\sqrt{2}}{2} i; w^{12} = 1$.

6. $1/z$ has absolute value $1/r$ and angle $-\theta; (1/r)e^{-i\theta}$ times $re^{i\theta}$ equals 1.

7. $A_1 - A_2$ gives complex matrix = vector multiplication $(A_1 + iA_2)(x_1 + ix_2) = b_1 + ib_2$.

8. The eigenvalues are $(1 + 3i)(1 - 3i) = 10.$

9. $2 + i; (2 + i)(1 + i) = 1 + 3i; e^{-i\pi/2} = -i; e^{-i\pi} = -1; \frac{1 - i}{1 + i} = -i; (-i)^{103} = i$.

10. $z + \overline{z}$ is real; $z - \overline{z}$ is pure imaginary; $z\overline{z}$ is positive; $z/\overline{z}$ has absolute value 1.

11. $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ includes $aI$ (which just adds $a$ to the eigenvalues and $b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. So the eigenvectors are $x_1 = (1, i)$ and $x_2 = (1, -i)$. The eigenvalues are $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$. We see $\overline{x_1} = x_2$ and $\overline{\lambda_1} = \lambda_2$ as expected for real matrices with complex eigenvalues.

12. (a) When $a = b = d = 1$ the square root becomes $\sqrt{4c}$; $\lambda$ is complex if $c < 0$ (b) $\lambda = 0$ and $\lambda = a + d$ when $ad = bc$ (c) the $\lambda$’s can be real and different.

13. Complex $\lambda$’s when $(a + d)^2 < 4(ad - bc)$; write $(a + d)^2 - 4(ad - bc)$ as $(a - d)^2 + 4bc$ which is positive when $bc > 0$.

14. The symmetric block matrix has real eigenvalues; so $i\lambda$ is real and $\lambda$ is pure imaginary.

15. (a) $2e^{i\pi/3}, 4e^{2i\pi/3}$ (b) $e^{2i\theta}, e^{4i\theta}$ (c) $7e^{3i\pi/2}, 49e^{3i\pi} (= -49)$ (d) $\sqrt{50}e^{-\pi i/4}, 50e^{-\pi i/2}$. 

Solutions to Exercises
16 \( r = 1 \), angle \( \frac{\pi}{2} - \theta \); multiply by \( e^{i\theta} \) to get \( e^{i\pi/2} = i \).

17 \( a + ib = 1, i, -1, -i, \pm \frac{1}{\sqrt{2}}, \pm \frac{i}{\sqrt{2}} \). The root \( w = w^{-1} = e^{-2\pi i/8} \) is \( 1/\sqrt{2} - i/\sqrt{2} \).

18 \( e^{2\pi i/3}, e^{4\pi i/3} \) are cube roots of 1. The cube roots of \( -1 \) are \( -1, e^{i\pi/3}, e^{-i\pi/3} \). Altogether six roots of \( z^6 = 1 \).

19 \( \cos 3\theta = \text{Re}[(\cos \theta + i \sin \theta)^3] = \cos^3 \theta - 3 \cos \theta \sin^2 \theta; \sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta \).

20 If the conjugate \( \overline{z} = 1/z \) then \( |z|^2 = 1 \) and \( z \) is any point \( e^{i\theta} \) on the unit circle.

21 \( e^i \) is at angle \( \theta = 1 \) on the unit circle; \( |i^e| = 1^e \); Infinitely many \( i^e = e^{(\pi/2 + 2\pi n)e} \).

22 (a) Unit circle (b) Spiral in to \( e^{-2\pi} \) (c) Circle continuing around to angle \( \theta = 2\pi^2 \).

Problem Set 9.2, page 443

1 \( \|u\| = \sqrt{9} = 3, \|v\| = \sqrt{3}, u^Hv = 3i + 2, v^Hu = -3i + 2 \) (this is the conjugate of \( u^Hv \)).

2 \( A^H A = \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & 1-i & 2 \end{bmatrix} \) and \( AA^H = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \) are Hermitian matrices. They share the eigenvalues 4 and 2.

3 \( z = \text{multiple of } (1+i, 1+i, -2); Az = 0 \) gives \( z^H A^H = 0^H \) so \( z \) (not \( \overline{z} \)) is orthogonal to all columns of \( A^H \) (using complex inner product \( z^H \) times columns of \( A^H \)).

4 The four fundamental subspaces are now \( C(A), N(A), C(A^H), N(A^H) \). \( A^H \text{ and not } A^T \).

5 (a) \( (A^H A)^H = A^H A \) again (b) If \( A^H Az = 0 \) then \( (z^H A^H)(Az) = 0 \). This is \( \|Az\|^2 = 0 \) so \( Az = 0 \). The nullspaces of \( A \) and \( A^H A \) are always the same.

6 (a) False \( A = Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) (b) True: \( -i \) is not an eigenvalue when \( S = S^H \).

7 \( cS \) is still Hermitian for real \( c \); \( (iS)^H = -iS^H = -iS \) is skew-Hermitian.
8 This $P$ is invertible and unitary. $P^2 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$, $P^3 = \begin{bmatrix} -i \\ -i \\ -i \end{bmatrix}$. Then $P^{100} = (-i)^3 P = -i P$. The eigenvalues of $P$ are the roots of $\lambda^3 = -i$, which are $i$ and $ie^{2\pi i/3}$ and $ie^{4\pi i/3}$.

9 One unit eigenvector is certainly $x_1 = (1, 1, 1)$ with $\lambda_1 = i$. The other eigenvectors are $x_2 = (1, w, w^2)$ and $x_3 = (1, w^2, w^4)$ with $w = e^{2\pi i/3}$. The eigenvector matrix is the Fourier matrix $F_3$. The eigenvectors of any unitary matrix like $P$ are orthogonal (using the correct complex form $x^H y$ of the inner product).

10 $(1, 1, 1), (1, e^{2\pi i/3}, e^{4\pi i/3}), (1, e^{4\pi i/3}, e^{2\pi i/3})$ are orthogonal (complex inner product!) because $P$ is an orthogonal matrix—and therefore its eigenvector matrix is unitary.

11 If $Q^H Q = I$ then $Q^{-1}(Q^H)^{-1} = Q^{-1}(Q^{-1})^H = I$ so $Q^{-1}$ is also unitary. Also $(QU)^H(QU) = U^H Q^H Q U = U^H U = I$ so $QU$ is unitary.

12 Determinant = product of the eigenvalues (all real). And $A = A^H$ gives $\det A = \overline{\det A}$.

13 $(x^H A^H)(Az) = \|Az\|^2$ is positive unless $Az = 0$. When $A$ has independent columns this means $z = 0$; so $A^H A$ is positive definite.

14 $S = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1+i \\ 1+i & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ -1-i & 1 \end{bmatrix}$. 

15 $K = (i A^T$ in Problem 14) $= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1-i \\ 1-i & 1 \end{bmatrix} \begin{bmatrix} 2i & 0 \\ 0 & -i \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ -1+i & 1 \end{bmatrix}$; $\lambda$’s are imaginary.

16 $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} \cos \theta + i \sin \theta & 0 \\ 0 & \cos \theta - i \sin \theta \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ has $|\lambda| = 1$.

17 $U = \frac{1}{L} \begin{bmatrix} 1 + \sqrt{3} & -1+i \\ 1+i & 1+\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{L} \begin{bmatrix} 1 + \sqrt{3} & 1-i \\ 1-i & 1+\sqrt{3} \end{bmatrix}$ with $L^2 = 6 + 2\sqrt{3}$. Unitary means $|\lambda| = 1$. $U = U^H$ gives real $\lambda$. Then trace zero gives $\lambda = 1$ and $-1$.

18 The $v$’s are columns of a unitary matrix $U$, so $U^H$ is $U^{-1}$. Then $z = UU^H z =$ (multiply by columns) $= v_1(v_1^H z) + \cdots + v_n(v_n^H z)$: a typical orthonormal expansion.
19 \[ z = (1, i, -2) \] completes an orthogonal basis for \( \mathbb{C}^3 \). So does any \( e^{i\theta}z \).

20 \( S = A + iB = (A + iB)^H = A^T - iB^T \); \( A \) is symmetric but \( B \) is skew-symmetric.

21 \( \mathbb{C}^n \) has dimension \( n \); the columns of any unitary matrix are a basis. For example use the columns of \( iI \):

\[
\begin{pmatrix}
i & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & i
\end{pmatrix}
\]

22 \( [ 1 ] \) and \( [ -1 ] \); any \( [ e^{i\theta} ] \):

\[
\begin{bmatrix}
a & b + ic \\
b - ic & d
\end{bmatrix}
\begin{bmatrix}
w & e^{i\theta}z \\
-w & e^{i\theta}w
\end{bmatrix}
\text{ with } |w|^2 + |z|^2 = 1
\]

and any angle \( \phi \).

23 The eigenvalues of \( A^H \) are complex conjugates of the eigenvalues of \( A \): \( \det(A - \lambda I) = 0 \) gives \( \det(A^H - \bar{\lambda}I) = 0 \).

24 \( (I - 2uu^H)^H = I - 2uu^H \) and also \( (I - 2uu^H)^2 = I - 4uu^H + 4u(u^Hu)u^H = I \).

The rank-1 matrix \( uu^H \) projects onto the line through \( u \).

25 Unitary \( U^HU = I \) means \( (A^T - iB^T)(A + iB) = (A^TA + B^TB) + i(A^TB - B^TA) = I \).

\( A^TA + B^TB = I \) and \( A^TB - B^TA = 0 \) which makes the block matrix orthogonal.

26 We are given \( A + iB = (A + iB)^H = A^T - iB^T \). Then \( A = A^T \) and \( B = -B^T \). So that

\[
\begin{bmatrix}
A & -B \\
B & A
\end{bmatrix}
\]

is symmetric.

27 \( SS^{-1} = I \) gives \( (S^{-1})^H S^H = I \). Therefore \( (S^{-1})^H = S^{-1} \) and \( S^{-1} \) is Hermitian.

28 If \( U \) has (complex) orthonormal columns, then \( U^HU = I \) and \( U \) is unitary. If those columns are eigenvectors of \( A \), then \( A = U\Lambda U^{-1} = U\Lambda U^H \) is normal. The direct test for a normal matrix (which is \( AA^H = A^HA \) because diagonals could be real!) and \( \Lambda^H \) surely commute:

\[
AA^H = (U\Lambda U^H)(U\Lambda^H U^H) = U(\Lambda\Lambda^H)U^H = U(A^H\Lambda)U^H = (U\Lambda^H U^H)(U\Lambda U^H) = A^HA.
\]

An easy way to construct a normal matrix is \( 1 + i \) times a symmetric matrix. Or take \( A = S + iT \) where the real symmetric \( S \) and \( T \) commute (Then \( A^H = S - iT \) and \( AA^H = A^HA \)).
Problem Set 9.3, page 450

1. Equation (3) (the FFT) is correct using $i^2 = -1$ in the last two rows and three columns.

2. $F^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & 1 & \frac{1}{2} \\ 1 & 1 & 1 & -1 \\ 1 & i^2 & -i & i \end{bmatrix} = \frac{1}{4} F^H$.

3. $F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & i \\ 1 & i^2 & -i & i \end{bmatrix}$ permutation last.

4. $D = \begin{bmatrix} e^{2\pi i/6} \hfill \\ e^{4\pi i/6} \end{bmatrix}$ (note 6 not 3) and $F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \end{bmatrix}$.

5. $F^{-1} w = v$ and $F^{-1} v = w/4$. Delta vector $\leftrightarrow$ all-ones vector.

6. $(F_4)^2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$ and $(F_4)^4 = 16I$. Four transforms recover the signal!

7. $c = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = Fc$. Also $C = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix} = FC$.

Adding $c + C$ gives $(1,1,1,1,1,1,1,1)$ to $(4,0,0,0) = 4$ (delta vector).

8. $c \rightarrow (1,1,1,0,0,0,0) \rightarrow (4,0,0,0,0,0,0) \rightarrow (4,0,0,0,4,0,0) = F_8 c$.

$C \rightarrow (0,0,0,1,1,1,1) \rightarrow (0,0,0,4,0,0,0) \rightarrow (4,0,0,0,-4,0,0,0) = F_8 C$.

9. If $w^{64} = 1$ then $w^2$ is a 32nd root of 1 and $\sqrt{w}$ is a 128th root of 1: Key to FFT.
Solutions to Exercises

10 For every integer \( n \), the \( n \)th roots of 1 add to zero. For even \( n \), they cancel in pairs. For any \( n \), use the geometric series formula \( 1 + w + \cdots + w^{n-1} = \frac{(w^n - 1)}{(w - 1)} = 0 \).

In particular for \( n = 3, 1 + (-1 + i\sqrt{3})/2 + (-1 - i\sqrt{3})/2 = 0 \).

11 The eigenvalues of \( P \) are \( 1, i, i^2 = -1 \), and \( i^3 = -i \). Problem 11 displays the eigenvectors. And also \( \det(P - \lambda I) = \lambda^4 - 1 \).

12 \( \Lambda = \text{diag}(1, i, i^2, i^3) \); 
\[
P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}
\] and \( P^T \) lead to \( \lambda^3 - 1 = 0 \).

13 \( e_1 = c_0 + c_1 + c_2 + c_3 \) and \( e_2 = c_0 + c_1 i + c_2 i^2 + c_3 i^3 \); \( E \) contains the four eigenvalues of \( C = F E F^{-1} \) because \( F \) contains the eigenvectors.

14 Eigenvalues \( e_1 = 2 - 1 - 1 = 0 \), \( e_2 = 2 - i - i^3 = 2 \), \( e_3 = 2 - (-1) - (-1) = 4 \), \( e_4 = 2 - i^3 - i^9 = 2 \). Just transform column 0 of \( C \). Check trace \( 0 + 2 + 4 + 2 = 8 \).

15 Diagonal \( E \) needs \( n \) multiplications, Fourier matrix \( F \) and \( F^{-1} \) need \( \frac{1}{2} n \log_2 n \) multiplications each by the FFT. The total is much less than the ordinary \( n^2 \) for \( C \) times \( x \).

16 The row \( 1, w^k, w^{2k}, \ldots \) in \( F \) is the same as the row \( 1, w^{N-k}, w^{N-2k}, \ldots \) in \( F \) because 
\[ w^{N-k} = e^{(2\pi i/N)(N-k)} = e^{2\pi i e^{-2\pi i/N}k} = 1 \text{ times } w^k. \] So \( F \) and \( F \) have the same rows in reversed order (except for row 0 which is all ones).

17 \[
\begin{array}{cccc}
0 & 000 & \text{reverses to} & 000 = 0 \\
1 & 001 & \text{reverses to} & 100 = 4 \\
2 & 010 & \text{reverses to} & 010 = 2 \quad \text{Now evens come before odds!} \\
3 & 011 & \text{reverses to} & 110 = 6 \\
4 & 100 & \text{reverses to} & 001 = 1 \\
5 & 101 & \text{reverses to} & 101 = 5 \\
6 & 110 & \text{reverses to} & 011 = 3 \\
7 & 111 & \text{reverses to} & 111 = 7
\end{array}
\]