Problem Set 8.1, page 407

1 With \( w = 0 \) linearity gives \( T(v + 0) = T(v) + T(0) \). Thus \( T(0) = 0 \). With \( c = -1 \) linearity gives \( T(-0) = -T(0) \). This is a second proof that \( T(0) = 0 \).

2 Combining \( T(cv) = cT(v) \) and \( T(dw) = dT(w) \) with addition gives \( T(cv + dw) = cT(v) + dT(w) \). Then one more addition gives \( cT(v) + dT(w) + eT(u) \).

3 (d) is not linear.

4 (a) \( S(T(v)) = v \) (b) \( S(T(v_1) + T(v_2)) = S(T(v_1)) + S(T(v_2)) \).

5 Choose \( v = (1, 1) \) and \( w = (-1, 0) \). Then \( T(v) + T(w) = (v + w) \) but \( T(v + w) = (0, 0) \).

6 (a) \( T(v) = v/\|v\| \) does not satisfy \( T(v + w) = T(v) + T(w) \) or \( T(cv) = cT(v) \) (b) and (c) are linear (d) satisfies \( T(cv) = cT(v) \).

7 (a) \( T(T(v)) = v \) (b) \( T(T(v)) = v + (2, 2) \) (c) \( T(T(v)) = -v \) (d) \( T(T(v)) = T(v) \).

8 (a) The range of \( T(v_1, v_2) = (v_1 - v_2, 0) \) is the line of vectors \( (c, 0) \). The nullspace is the line of vectors \( (c, c) \). (b) \( T(v_1, v_2, v_3) = (v_1, v_2) \) has Range \( \mathbb{R}^2 \), kernel \( \{(0, 0, v_3)\} \) (c) \( T(v) = 0 \) has Range \( \{0\} \), kernel \( \mathbb{R}^2 \) (d) \( T(v_1, v_2) = (v_1, v_1) \) has Range = multiples of \( (1, 1) \), kernel = multiples of \( (1, -1) \).

9 If \( T(v_1, v_2, v_3) = (v_2, v_3, v_1) \) then \( T(T(v)) = (v_3, v_1, v_2) \), \( T^3(v) = v \), \( T^{100}(v) = T(v) \).

10 (a) \( T(1, 0) = 0 \) (b) \( (0, 0, 1) \) is not in the range (c) \( T(0, 1) = 0 \).

11 For multiplication \( T(v) = Av \): \( V = \mathbb{R}^n \), \( W = \mathbb{R}^m \); the outputs fill the column space; \( v \) is in the kernel if \( Av = 0 \).

12 \( T(v) = (4, 4); (2, 2); (2, 2) \); if \( v = (a, b) = b(1, 1) + \frac{a-b}{2}(2, 0) \) then \( T(v) = b(2, 2) + (0, 0) \).

13 The distributive law (page 69) gives \( A(M_1 + M_2) = AM_1 + AM_2 \). The distributive law over \( c \)'s gives \( A(cM) = c(AM) \).
This $A$ is invertible. Multiply $AM = 0$ and $AM = B$ by $A^{-1}$ to get $M = 0$ and $M = A^{-1}B$. The kernel contains only the zero matrix $M = 0$.

This $A$ is not invertible. $AM = I$ is impossible.

No matrix $A$ gives $A \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. To professors: Linear transformations on matrix space come from 4 by 4 matrices. Those in Problems 13–15 were special.

For $T(M) = MT$ (a) $T^2 = I$ is True (b) True (c) True (d) False.

$T(I) = 0$ but $M = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = T(M)$; these $M$'s fill the range. Every $M = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ is in the kernel. Notice that dim (range) + dim (kernel) = 3 + 1 = dim (input space of 2 by 2 $M$'s).

$T(T^{-1}(M)) = M$ so $T^{-1}(M) = A^{-1}MB^{-1}$.

(a) Horizontal lines stay horizontal, vertical lines stay vertical (b) House squashes onto a line (c) Vertical lines stay vertical because $T(1, 0) = (a_{11}, 0)$.

$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ doubles the width of the house. $A = \begin{bmatrix} .7 & .7 \\ .3 & .3 \end{bmatrix}$ projects the house (since $A^2 = A$ from trace = 1 and $\lambda = 0, 1$). The projection is onto the column space of $A = \text{line through (.7, .3)}$. $U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ will shear the house horizontally: The point at $(x, y)$ moves over to $(x + y, y)$.

(a) $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ with $d > 0$ leaves the house $AH$ sitting straight up (b) $A = 3I$ expands the house by 3 (c) $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ rotates the house.

$T(v) = -v$ rotates the house by $180^\circ$ around the origin. Then the affine transformation $T(v) = -v + (1, 0)$ shifts the rotated house one unit to the right.

A code to add a chimney will be gratefully received!
25 This code needs a correction: add spaces between $-10\ 10\ -10\ 10$

26 The matrix $\begin{bmatrix} 1 & 0 \\ 0 & .1 \end{bmatrix}$ compresses vertical distances by 10 to 1. The matrix $\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ projects onto the 45° line. The matrix $\begin{bmatrix} .5 & .5 \\ -.5 & .5 \end{bmatrix}$ rotates by 45° clockwise and contracts by a factor of $\sqrt{2}$ (the columns have length $1/\sqrt{2}$). The matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has determinant $-1$ so the house is “flipped and sheared.” One way to see this is to factor the matrix as $LDL^T$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \text{(shear)} \text{ (flip left-right)} \text{ (shear)}.$$

27 Also 30 emphasizes that circles are transformed to ellipses (see figure in Section 6.7).

28 A code that adds two eyes and a smile will be included here with public credit given!

29 (a) $ad - bc = 0$ (b) $ad - bc > 0$ (c) $|ad - bc| = 1$. If vectors to two corners transform to themselves then by linearity $T = I$. (Fails if one corner is $(0, 0)$.)

30 Linear transformations keep straight lines straight! And two parallel edges of a square (edges differing by a fixed $v$) go to two parallel edges (edges differing by $T(v)$). So the output is a parallelogram.

Problem Set 8.2, page 418

For $Sv = d^2v/dx^2$

1 $v_1, v_2, v_3, v_4 = 1, x, x^2, x^3$

The matrix for $S$ is $B = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

2 $Sv = d^2v/dx^2 = 0$ for linear functions $v(x) = a + bx$. All $(a, b, 0, 0)$ are in the nullspace of the second derivative matrix $B$.

3 $(\text{Matrix } A)^2 = B$ when (transformation $T)^2 = S$ and output basis = input basis.
4 The third derivative matrix has 6 in the (1, 4) position; since the third derivative of \( x^3 \) is 6. This matrix also comes from \( AB \). The fourth derivative of a cubic is zero, and \( B^2 \) is the zero matrix.

5 \( T(v_1 + v_2 + v_3) = 2w_1 + w_2 + 2w_3; A \) times (1, 1, 1) gives (2, 1, 2).

6 \( v = c(v_2 - v_3) \) gives \( T(v) = 0 \); nullspace is (0, c, −c); solutions (1, 0, 0) + (0, c, −c).

7 (1, 0, 0) is not in the column space of the matrix \( A \), and \( w_1 \) is not in the range of the linear transformation \( T \). Key point: Column space of matrix matches range of transformation.

8 We don’t know \( T(w) \) unless the \( w \)’s are the same as the \( v \)’s. In that case the matrix is \( A^2 \).

9 Rank of \( A = 2 \) = dimension of the range of \( T \). The outputs \( Av \) (column space) match the outputs \( T(v) \) (the range of \( T \)). The “output space” \( W \) is like \( \mathbb{R}^m \); it contains all outputs but may not be filled up.

10 The matrix for \( T \) is \( A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \). For the output \( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \) choose input \( v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \).

\[ A^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \] This means: For the output \( w_1 \) choose the input \( v_1 - v_2 \).

11 \( A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \) so \( T^{-1}(w_1) = v_1 - v_2, T^{-1}(w_2) = v_2 - v_3, T^{-1}(w_3) = v_3 \). The columns of \( A^{-1} \) describe \( T^{-1} \) from \( W \) back to \( V \). The only solution to \( T(v) = 0 \) is \( v = 0 \).

12 (c) \( T^{-1}(T(w_1)) = w_1 \) is wrong because \( w_1 \) is not generally in the input space.

13 (a) \( T(v_1) = v_2, T(v_2) = v_1 \) is its own inverse (b) \( T(v_1) = v_1, T(v_2) = 0 \) has \( T^2 = T \) (c) If \( T^2 = I \) for part (a) and \( T^2 = T \) for part (b), then \( T \) must be \( I \).
Solutions to Exercises

14 (a) \[
\begin{bmatrix}
2 & 1 \\
5 & 3
\end{bmatrix}
\] (b) \[
\begin{bmatrix}
3 & -1 \\
-5 & 2
\end{bmatrix}
\] = inverse of (a) (c) \[
\begin{bmatrix}
2 \\
6
\end{bmatrix}
\] must be \[
2 \begin{bmatrix}
1 \\
3
\end{bmatrix}
\].

15 (a) \[
M = \begin{bmatrix}
r & s \\
t & u
\end{bmatrix}
\]
transforms \[
\begin{bmatrix} 1 \\ 0 \end{bmatrix}
\] and \[
\begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]
to \[
\begin{bmatrix} r \\ t \end{bmatrix}
\] and \[
\begin{bmatrix} s \\ u \end{bmatrix}
\]; this is the “easy” direction. (b) \[
N = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}^{-1}
\]
transforms in the inverse direction, back to the standard basis vectors. (c) \(ad = bc\) will make the forward matrix singular and the inverse impossible.

16 \[
MW = \begin{bmatrix}
1 & 0 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
2 & 1 \\
5 & 3
\end{bmatrix}^{-1}
= \begin{bmatrix}
3 & -1 \\
-7 & 3
\end{bmatrix}.
\]

17 Recording basis vectors is done by a Permutation matrix. Changing lengths is done by a positive diagonal matrix.

18 \((a, b) = (\cos \theta, -\sin \theta)\). Minus sign from \(Q^{-1} = Q^T\).

19 \[
M = \begin{bmatrix}
1 & 1 \\
4 & 5
\end{bmatrix}; \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix} = \text{first column of } M^{-1} = \text{coordinates of } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ in basis}
\begin{bmatrix}
1 \\
4
\end{bmatrix}.
\]

20 \(w_2(x) = 1 - x^2; w_3(x) = \frac{1}{2}(x^2 - x); y = 4w_1 + 5w_2 + 6w_3\).

21 \(w\)’s to \(v\)’s: \[
\begin{bmatrix}
0 & 1 & 0 \\
.5 & 0 & -.5 \\
.5 & -1 & .5
\end{bmatrix}
\] . \(v\)’s to \(w\)’s: inverse matrix = \[
\begin{bmatrix}
1 & 0 & 0 \\
1 & -1 & 1
\end{bmatrix}
\]. The key idea: The matrix multiplies the coordinates in the \(v\) basis to give the coordinates in the \(w\) basis.

22 The 3 equations to match \(4, 5, 6\) at \(x = a, b, c\) are \[
\begin{bmatrix}
1 & a & a^2 \\
1 & b & b^2 \\
1 & c & c^2
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C
\end{bmatrix}
= \begin{bmatrix}
4 \\
5 \\
6
\end{bmatrix}.
\] This Vandermonde determinant equals \((b - a)(c - a)(c - b)\). So \(a, b, c\) must be distinct to have \(\det \neq 0\) and one solution \(A, B, C\).
23 The matrix $M$ with these nine entries must be invertible.

24 Start from $A = QR$. Column 2 is $a_2 = r_{12}q_1 + r_{22}q_2$. This gives $a_2$ as a combination of the $q$'s. So the change of basis matrix is $R$.

25 Start from $A = LU$. Row 2 of $A$ is $\ell_{21}(row 1 of U) + \ell_{22} (row 2 of U)$. The change of basis matrix is always invertible, because basis goes to basis.

26 The matrix for $T(\mathbf{v}_1) = \lambda_i \mathbf{v}_1$ is $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$.

27 If $T$ is not invertible, $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ is not a basis. We couldn’t choose $\mathbf{w}_1 = T(\mathbf{v}_1)$.

28 (a) $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ gives $T(\mathbf{v}_1) = 0$ and $T(\mathbf{v}_2) = 3\mathbf{v}_1$. (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ gives $T(\mathbf{v}_1) = \mathbf{v}_1$ and $\bar{T}(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{v}_1$ (which combine into $T(\mathbf{v}_2) = 0$ by linearity).

29 $T(x, y) = (x, -y)$ is reflection across the $x$-axis. Then reflect across the $y$-axis to get $S(x, -y) = (-x, -y)$. Thus $ST = -I$.

30 $S$ takes $(x, y)$ to $(-x, y)$. $S(T(\mathbf{v})) = (-1, 2)$. $S(\mathbf{v}) = (-2, 1)$ and $T(S(\mathbf{v})) = (1, -2)$.

31 Multiply the two reflections to get $\begin{bmatrix} \cos 2(\theta - \alpha) & -\sin 2(\theta - \alpha) \\ \sin 2(\theta - \alpha) & \cos 2(\theta - \alpha) \end{bmatrix}$ which is rotation by $2(\theta - \alpha)$. In words: $(1, 0)$ is reflected to have angle $2\alpha$, and that is reflected again to angle $2\theta - 2\alpha$.

32 The matrix for $T$ in this basis is $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

33 Multiplying by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ gives $T(\mathbf{v}_1) = A \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ c \\ 0 \end{bmatrix} = a\mathbf{v}_1 + c\mathbf{v}_3$. Similarly $T(\mathbf{v}_2) = a\mathbf{v}_2 + c\mathbf{v}_4$ and $T(\mathbf{v}_3) = b\mathbf{v}_1 + d\mathbf{v}_3$ and $T(\mathbf{v}_4) = b\mathbf{v}_2 + d\mathbf{v}_4$. The matrix for $T$ in this basis is $\begin{bmatrix} a & b & 0 & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}$.

34 False: We will not know $T(\mathbf{v})$ for energy $\mathbf{v}$ unless the $n\mathbf{v}$’s are linearly independent.
Problem Set 8.3, page 429

1. For this matrix $J$, the rank of $J - 3I$ is 3 so the dimension of the nullspace is only 1. There is only 1 independent eigenvector even though $\lambda = 3$ is a double root of $\det(J - \lambda I) = 0$: a repeated eigenvalue.

\[
J = \begin{bmatrix}
2 \\
3 \\
3
\end{bmatrix}.
\]

2. $J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is similar to all other 2 by 2 matrices $A$ that have 2 zero eigenvalues but only 1 independent eigenvector. Then $J = B_1^{-1}A_1B_1$ is the same as $B_1J = A_1B_1:

\[
B_1J = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} = A_1B_1
\]

\[
B_2J = \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -8 \\ 2 & -4 \end{bmatrix} = A_2B_2
\]

3. Every matrix is similar to its transpose (same eigenvalues, same multiplicity, more than that the same Jordan form). In this example

\[
BJ = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} = J^T B.
\]

4. Here $J$ and $K$ are different Jordan forms (block sizes 2, 2 versus block sizes 3, 1). Even though $J$ and $K$ have the same $\lambda$’s (all zero) and same rank, $J$ and $K$ are not similar.

If $BK = JB$ then $B$ is not invertible:

\[
BK = B \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ 0 & b_{31} & b_{32} & 0 \\ 0 & b_{41} & b_{42} & 0 \end{bmatrix}
\]
$J B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} b_{21} & b_{22} & b_{23} & b_{24} \\ 0 & 0 & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Those right hand sides agree only if $b_{21} = 0, b_{41} = 0, b_{24} = 0, b_{44} = 0, b_{22} = 0, b_{42} = 0$. But then also $b_{11} = b_{22} = 0$ and $b_{31} = b_{42} = 0$. So the first column has $b_{11} = b_{21} = b_{31} = b_{41} = 0$ and $B$ is not invertible.

5 If $A^3$ is the zero matrix then every eigenvalue of $A$ is $\lambda = 0$ (because $A x = \lambda x$ leads to $\theta = A^3 x = \lambda^3 x$). The Jordan form $J$ will also have $J^3 = 0$ because $J = B^{-1} A B$ has $J^3 = B^{-1} A^3 B = 0$. The blocks of $J$ must become zero blocks in $J^3$. So those blocks of $J$ can be

$$\begin{bmatrix} 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{but not} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \text{third power} \\ \text{is not zero} \end{pmatrix}$$

The rank of $J$ (and $A$) is largest if every block is 3 by 3 of rank 2. Then rank $\leq \frac{2}{3} n$.

If $A^n = \text{zero matrix}$ then $A$ is not invertible and rank $(A) < n$.

6 This question substitutes $u_1 = t e^{\lambda t}$ and $u_2 = e^{\lambda t}$ to show that $u_1, u_2$ solve the system $u' = J u$:

$$u_1' = \lambda u_1 + u_2 \quad e^{\lambda t} + t \lambda e^{\lambda t} = \lambda (t e^{\lambda t}) + (e^{\lambda t})$$

$$u_2' = \lambda u_2 \quad \lambda e^{\lambda t} = \lambda (e^{\lambda t}).$$

Certainly $u_1 = 0$ and $u_2 = 1$ at $t = 0$, so we have the solution and it involves $t e^{\lambda t}$ (the factor $t$ appears because $\lambda$ is a double eigenvalue of $J$).

7 The equation $u_{k+2} - 2 \lambda u_{k+1} + \lambda^2 u_k$ is certainly solved by $u_k = \lambda^k$. But this is a second order equation and there must be another solution. In analogy with $t e^{\lambda t}$ for the differential equation in 8.3.6, that second solution is $u_k = k \lambda^k$. Check:
\[(k + 2)\lambda^{k+2} - 2\lambda(k + 1)\lambda^{k+1} + \lambda^2(k)\lambda^k = [k + 2 - 2(k + 1) + k]\lambda^{k+2} = 0.\]

8 \(\lambda^3 = 1\) has 3 roots \(\lambda = 1\) and \(e^{2\pi i/3}\) and \(e^{4\pi i/3}\). Those are \(1, \lambda, \lambda^2\) if we take \(\lambda = e^{2\pi i/3}\). The Fourier matrix is

\[
F_3 = \begin{bmatrix}
1 & 1 & 1 \\
1 & \lambda & \lambda^2 \\
1 & \lambda^2 & \lambda^4
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 \\
1 & e^{2\pi i/3} & e^{4\pi i/3} \\
1 & e^{4\pi i/3} & e^{8\pi i/3}
\end{bmatrix}.
\]

9 A 3 by 3 circulant matrix has the form on page 425:

\[
C = \begin{bmatrix}
c_0 & c_1 & c_2 \\
c_2 & c_0 & c_1 \\
c_1 & c_2 & c_0
\end{bmatrix}
\]

with \(C = (c_0 + c_1 + c_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\) and \(C = (c_0 + c_1\lambda + c_2\lambda^2) \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}\) and \(C = (c_0 + c_1\lambda^2 + c_2\lambda^4) \begin{bmatrix} 1 \\ \lambda^2 \\ \lambda^4 \end{bmatrix}\).

Those 3 eigenvalues of \(C\) are exactly the 3 components of \(Fe = F \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}\).

10 The Fourier cosine coefficient \(c_3\) is in formula (7) with integrals from \(-\pi\) to \(\pi\). Because \(f\) drops to zero at \(x = L\), the integral stops at \(L\):

\[
a_3 = \frac{\int_{-L}^{L} f(x) \cos 3x \, dx}{\int_{-L}^{L} (\cos 3x)^2 \, dx} = \frac{1}{\pi} \int_{-L}^{L} (1) \cos 3x \, dx = \frac{1}{3\pi} \left[ \frac{\sin 3x}{3} \right]_{x=-L}^{x=L} = \frac{2 \sin 3L}{3\pi}.
\]

Note that we should have defined \(f(x) = 0\) for \(L < |x| < \pi\) (not \(2\pi\)).