Problem Set 5.1, page 254

1 \[ \det(2A) = 2^4 \det A = 8; \det(-A) = (-1)^4 \det A = \frac{1}{2}; \det(A^2) = \frac{1}{4}; \det(A^{-1}) = 2. \]

2 \[ \det\left(\frac{1}{2}A\right) = \left(\frac{1}{2}\right)^3 \det A = -\frac{1}{8} \quad \text{and} \quad \det(-A) = (-1)^3 \det A = 1; \det(A^2) = 1; \] \[ \det(A^{-1}) = -1. \]

3 (a) False: \( \det(I + I) \) is not \( 1 + 1 \) (except when \( n = 1 \))

(b) True: The product rule extends to \( ABC \) (use it twice)

(c) False: \( \det(4A) \) is \( 4^n \det A \)

(d) False: \( \det(A) = 0 \)

4 Exchange rows 1 and 3 to show \( |J_3| = -1 \). Exchange rows 1 and 4, then rows 2 and 3 to show \( |J_4| = 1 \).

5 \( |J_5| = 1 \) by exchanging row 1 with 5 and row 2 with 4. \( |J_6| = -1, \quad |J_7| = -1. \) Determinants 1, 1, -1, -1 repeat in cycles of length 4 so the determinant of \( J_{101} \) is +1.

6 To prove Rule 6, multiply the zero row by \( t = 2 \). The determinant is multiplied by 2 (Rule 3) but the matrix is the same. So \( 2 \det(A) = \det(A) \) and \( \det(A) = 0 \).

7 \( \det(Q) = 1 \) for rotation and \( \det(Q) = 1 - 2 \sin^2 \theta - 2 \cos^2 \theta = -1 \) for reflection.

8 \( Q^TQ = I \Rightarrow |Q^T|^{|Q|} = |Q|^2 = 1 \Rightarrow |Q| = \pm 1; \) \( Q^n \) stays orthogonal so its determinant can’t blow up as \( n \to \infty \).

9 \( \det A = 1 \) from two row exchanges, \( \det B = 2 \) (subtract rows 1 and 2 from row 3, then columns 1 and 2 from column 3). \( \det C = 0 \) (equal rows) even though \( C = A + B! \)

10 If the entries in every row add to zero, then \((1, 1, \ldots, 1)\) is in the nullspace: singular \( A \) has \( \det = 0 \). (The columns add to the zero column so they are linearly dependent.)

If every row adds to one, then rows of \( A - I \) add to zero (not necessarily \( \det A = 1 \)).

11 \( CD = -DC \Rightarrow \det CD = (-1)^n \det DC \) and not just \( -\det DC \). If \( n \) is even then \( \det CD = \det DC \) and we can have an invertible \( CD \).

12 \( \det(A^{-1}) \) divides twice by \( ad - bc \) (once for each row). This gives \( \det A^{-1} = \frac{ad - bc}{(ad - bc)^2} = \frac{1}{ad - bc} \).
13 Pivots 1, 1, 1 give determinant = 1; pivots 1, −2, −3/2 give determinant = 3.

14 \(\text{det}(A) = 36\) and the 4 by 4 second difference matrix has \(\text{det} = 5\).

15 The first determinant is 0, the second is \(1 - 2T^2 + T^4 = (1 - T^2)^2\).

16 A singular rank one matrix has determinant = 0. The skew-symmetric \(K\) also has \(\text{det} K = 0\) (see #17): a skew-symmetric matrix \(K\) of odd order 3.

17 Any 3 by 3 skew-symmetric \(K\) has \(\text{det}(K^T) = \text{det}(-K) = (-1)^3\text{det}(K)\). This is \(-\text{det}(K)\). But always \(\text{det}(K^T) = \text{det}(K)\). So we must have \(\text{det}(K) = 0\) for 3 by 3.

\[
\begin{vmatrix}
1 & a & a^2 \\
b & b^2 & \\
c & c^2 & \\
\end{vmatrix}
= \begin{vmatrix}
1 & a & a^2 \\
0 & b - a & b^2 - a^2 \\
0 & c - a & c^2 - a^2 \\
\end{vmatrix}
\] (to reach 2 by 2, eliminate \(a\) and \(a^2\) in row 1 by column operations)—subtract \(a\) and \(a^2\) times column 1 from columns 2 and 3. Factor out \(b - a\) and \(c - a\) from the 2 by 2:

\[
(b - a)(c - a)
\begin{vmatrix}
1 & b + a \\
1 & c + a \\
\end{vmatrix}
= (b - a)(c - a)(c - b).
\]

18 For triangular matrices, just multiply the diagonal entries: \(\text{det}(U) = 6\), \(\text{det}(U^{-1}) = \frac{1}{6}\), and \(\text{det}(U^2) = 36\). 2 by 2 matrix: \(\text{det}(U) = ad\), \(\text{det}(U^2) = a^2d^2\). If \(ad \neq 0\) then \(\text{det}(U^{-1}) = 1/ad\).

19 \(\left|\begin{array}{cc}
a - Lc & b - Ld \\
c - La & d - Lb \\
\end{array}\right|\) reduces to \((ad - bc)(1 - L\ell)\). The determinant changes if you do two row operations at once.

20 We can exchange rows using the three elimination steps in the problem, followed by multiplying row 1 by −1. So Rules 5 and 3 give Rule 2. (Since Rules 4 and 3 give 5, they also give Rule 2.)

21 \(\text{det}(A) = 3\), \(\text{det}(A^{-1}) = \frac{1}{3}\), \(\text{det}(A - \lambda I) = \lambda^2 - 4\lambda + 3\). The numbers \(\lambda = 1\) and \(\lambda = 3\) give \(\text{det}(A - \lambda I) = 0\). The (singular !) matrices are

\[
A - \lambda I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\text{and}
A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.
\]
Solutions to Exercises

Note to instructor: You could explain that this is the reason determinants come before eigenvalues. Identify \( \lambda = 1 \) and \( \lambda = 3 \) as the eigenvalues of \( A \).

**23** \[
A = \begin{bmatrix}
4 & 1 \\
2 & 3
\end{bmatrix}
\] has \( \det(A) = 10 \), \( A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix} \), \( \det(A^2) = 100 \), \( A^{-1} = \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} \) has \( \det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = 0 \) when \( \lambda = 2 \) or 5; those are eigenvalues.

**24** Here \( A = LU \) with \( \det(L) = 1 \) and \( \det(U) = -6 \) = product of pivots, so also \( \det(A) = -6 \). \( \det(U^{-1}L^{-1}) = -\frac{1}{6} = 1/\det(A) \) and \( \det(U^{-1}L^{-1}A) = \det(I) = 1 \).

**25** When the \( i, j \) entry is \( ij \), row 2 = 2 times row 1 so \( \det(A) = 0 \).

**26** When the \( ij \) entry is \( i + j \), row 3 - row 2 = row 2 - row 1 so \( A \) is singular: \( \det(A) = 0 \).

**27** \( \det(A) = abc \), \( \det(B) = -abcd \), \( \det(C) = a(b - a)(c - b) \) by doing elimination.

**28** (a) True: \( \det(AB) = \det(A) \det(B) = 0 \) (b) False: A row exchange gives \( -\det = \) product of pivots. (c) False: \( A = 2I \) and \( B = I \) have \( A - B = I \) but the determinants have \( 2^n - 1 \neq 1 \) (d) True: \( \det(AB) = \det(A) \det(B) = \det(BA) \).

**29** \( A \) is rectangular so \( \det(A^T A) \neq (\det A^T)(\det A) \): these determinants are not defined.

In fact if \( A \) is tall and thin \( (m > n) \), then \( \det(A^T A) \) adds up \( |\det B|^2 \) where the \( B \)'s are all the \( n \) by \( n \) submatrices of \( A \).

**30** Derivatives of \( f = \ln(ad - bc) \):

\[
\begin{bmatrix}
\frac{\partial f}{\partial a} & \frac{\partial f}{\partial c} \\
\frac{\partial f}{\partial b} & \frac{\partial f}{\partial d}
\end{bmatrix} = \begin{bmatrix}
\frac{d}{ad - bc} & -\frac{b}{ad - bc} \\
-\frac{c}{ad - bc} & \frac{a}{ad - bc}
\end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix} = A^{-1}.
\]

**31** The Hilbert determinants are \( 1, 8 \times 10^{-2}, 4.6 \times 10^{-4}, 1.6 \times 10^{-7}, 3.7 \times 10^{-12}, 5.4 \times 10^{-18}, 4.8 \times 10^{-25}, 2.7 \times 10^{-33}, 9.7 \times 10^{-43}, 2.2 \times 10^{-53} \). Pivots are ratios of determinants so the 10th pivot is near \( 10^{-10} \). The Hilbert matrix is numerically difficult \((ill-conditioned)\). Please see the Figure 7.1 and Section 8.3.
Typical determinants of $\text{rand}(n)$ are $10^6, 10^{25}, 10^{79}, 10^{218}$ for $n = 50, 100, 200, 400$. $\text{randn}(n)$ with normal distribution gives $10^{31}, 10^{78}, 10^{186}, \text{Inf}$ which means $\geq 2^{1024}$. MATLAB allows $1.99999999999999 \times 2^{1023} \approx 1.8 \times 10^{308}$ but one more 9 gives $\text{Inf}!$

I now know that maximizing the determinant for $1, -1$ matrices is Hadamard’s problem (1893): see Brenner in American Math. Monthly volume 79 (1972) 626-630. Neil Sloane’s wonderful On-Line Encyclopedia of Integer Sequences (research.att.com/~njas) includes the solution for small $n$ (and more references) when the problem is changed to $0, 1$ matrices. That sequence A003432 starts from $n = 0$ with 1, 1, 1, 2, 3, 5, 9. Then the 1, -1 maximum for size $n$ is $2^{n-1}$ times the 0, 1 maximum for size $n - 1$ (so (32)(5) = 160 for $n = 6$ in sequence A003433).

To reduce the 1, -1 problem from 6 by 6 to the 0, 1 problem for 5 by 5, multiply the six rows by $\pm 1$ to put +1 in column 1. Then subtract row 1 from rows 2 to 6 to get a 5 by 5 submatrix $S$ with entries $-2$ and 0. Then divide $S$ by $-2$.

Here is an advanced MATLAB code that finds a 1, -1 matrix with largest $\det A = 48$ for $n = 5$:

```matlab
n = 5; p = (n - 1)^2; A0 = ones(n); maxdet = 0;
for k = 0 : 2^p - 1
    Asub = rem(floor(k * 2.^(p+1 : 0)), 2); A = A0; A(2 : n, 2 : n) = 1 - 2*Asub;
    reshape(Asub, n - 1, n - 1);
    if abs(det(A)) > maxdet, maxdet = abs(det(A)); maxA = A; end
end

Output: maxA =

```

```
1 1 1 1 1
1 1 1 -1 -1
1 1 -1 1 -1
1 -1 1 1 -1
1 -1 -1 -1 1
```

maxdet = 48.
34 Reduce $B$ by row operations to \[ \text{row 3; row 2; row 1}. \] Then $\det B = -6$ (odd permutation from the order of the rows in $A$).

**Problem Set 5.2, page 266**

1 $\det A = 1 + 18 + 12 - 9 - 4 - 6 = 12$, the rows of $A$ are independent; $\det B = 0$, row 1 + row 2 = row 3 so the rows of $B$ are linearly dependent; $\det C = -1$, so $C$ has independent rows ($\det C$ has one term, an odd permutation).

2 $\det A = -2$, independent; $\det B = 0$, dependent; $\det C = -1$, independent but $\det D = 0$ because its submatrix $B$ has dependent rows.

3 The problem suggests 3 ways to see that $\det A = 0$: All cofactors of row 1 are zero. $A$ has rank $\leq 2$. Each of the 6 terms in $\det A$ is zero. Notice also that column 2 has no pivot.

4 $a_{11}a_{23}a_{32}a_{44}$ gives $-1$, because the terms $a_{23}a_{32}$ have columns 2 and 3 in reverse order. $a_{14}a_{23}a_{32}a_{41}$ which has 2 row exchanges gives $+1$, $\det A = 1 - 1 = 0$. Using the same entries but now taken from $B$, $\det B = 2 \cdot 4 \cdot 4 \cdot 2 - 1 \cdot 4 \cdot 4 \cdot 1 = 64 - 16 = 48$.

5 Four zeros in the same row guarantee $\det = 0$ (and also four zeros in the same column).

6 (a) If $a_{11} = a_{22} = a_{33} = 0$ then 4 terms will be zeros (b) 15 terms must be zero. Effectively we are counting the permutations that make everyone move; 2, 3, 1 and 3, 1, 2 for $n = 3$ mean that the other 4 permutations take a term from the diagonal of $A$; so those terms are 0 when the diagonal is all zeros.

7 $5!/2 = 60$ permutation matrices (half of $5! = 120$ permutations) have $\det = +1$. Move row 5 of $I$ to the top; then starting from (5, 1, 2, 3, 4) elimination will do four row exchanges on $P$.

8 If $\det A \neq 0$, then certainly some term $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ in the big formula is not zero! Move rows 1, 2, \ldots, $n$ into rows $\alpha, \beta, \ldots, \omega$. Then all these nonzero $a$’s will be on the main diagonal.
The big formula has six terms all ±1: say q are −1 and 6 − q are 1. Then \( \det A = -q + 6 - q = \text{even} \) (so \( \det A = 5 \) is impossible). Also \( \det A = 6 \) is impossible. All 3 even permutations like \( a_{11}a_{22}a_{33} \) would have to give +1 (so an even number of −1’s in the matrix). But all 3 odd permutations like \( a_{11}a_{23}a_{32} \) would have to give −1 (so an odd number of −1’s in the matrix). We can’t have it both ways, so \( \det A = 4 \) is best possible and not hard to arrange: put −1’s on the main diagonal.

The 4!/2 = 12 even permutations are \( (1, 2, 3, 4), (2, 1, 4, 3), (3, 1, 4, 2), (4, 3, 2, 1), \) and 8 \( P \)'s with one number in place and even permutation of the other three numbers: examples are 1, 3, 4, 2 and 1, 4, 2, 3.

\[ \det(I + P_{\text{even}}) \] is always 16 or 4 or 0 (16 comes from \( I + I \)).

\[ \det B = 1(0) + 2(42) + 3(-35) = -21. \]

\[ \det B = 1(0) + 2(42) + 3(-35) = -21. \]

Puzzle: \( \det D = 441 = (-21)^2 \). Why is \( \det(\text{cofactor matrix}) = (\det \text{ matrix})^{n-1} \)?

\[ C = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \]

\[ D = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix} \]

Therefore \( A^{-1} = \frac{1}{4} C^T = C^T / \det A. \)

\[ A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \]

\[ A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \]

\[ C_1 = 0, \ C_2 = -1, \ C_3 = 0, \ C_4 = 1 \]

\[ \text{by cofactors of row 1 then cofactors of column 1. Therefore } C_{10} = -C_8 = C_6 = -C_4 = C_2 = -1. \]

For the matrices in Problem 13 to produce nonzeros in the big formula, we must choose 1’s from column 2 then column 1, column 4 then column 3, and so on. Therefore \( n \) must be even to have \( \det \neq 0 \). The number of row exchanges is \( n/2 \) so the overall determinant is \( C_n = (-1)^{n/2} \).

The 1, 1 cofactor of the \( n \) by \( n \) matrix is \( E_{n-1} \). The 1, 2 cofactor has a single 1 in its first column, with cofactor \( E_{n-2} \): sign gives −\( E_{n-2} \). So \( E_n = E_{n-1} - E_{n-2} \). Then \( E_1 \) to \( E_6 \) is 1, 0, −1, −1, 0, 1 and this cycle of six will repeat: \( E_{100} = E_4 = -1. \)

The 1, 1 cofactor of the \( n \) by \( n \) matrix is \( F_{n-1} \). The 1, 2 cofactor has a 1 in column 1, with cofactor \( F_{n-2} \). Multiply by \( (-1)^{1+2} \) and also \( (-1) \) from the 1, 2 entry to find \( F_n = F_{n-1} + F_{n-2} \). So these determinants are Fibonacci numbers.
17 Use cofactors along row 4 instead of row 1 (last row instead of first).
\[ |B_4| = 2 \det \begin{bmatrix} 1 & -1 \\ -1 & 2 & -1 \\ -1 & 2 \\ -1 & 2 \end{bmatrix} + \det \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = 2|B_3| - \det \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}. \]
So \(|B_4| = 2|B_3| - |B_2|\).

18 Rule 3 (linearity in row 1) gives \(|B_n| = |A_n| - |A_{n-1}| = (n + 1) - n = 1.\)

19 Since \(x, x^2, x^3\) are all in the same row, they never multiply each other in \(\det V_4\). The determinant is zero at \(x = a\) or \(b\) or \(c\) because of equal rows! So \(\det V\) has factors \((x - a)(x - b)(x - c)\). Multiply by the cofactor \(V_3\). The Vandermonde matrix \(V_{ij} = (x_i)^{j-1}\) is for fitting a polynomial \(p(x) = b\) at the points \(x_i\). It has \(\det V = \text{product of all } x_k - x_m \text{ for } k > m\).

20 \(G_2 = -1, G_3 = 2, G_4 = -3,\) and \(G_n = (-1)^{n-1}(n - 1)\). One way to reach that \(G_n\) is to multiply the \(n\) eigenvalues \(-1, -1, \ldots, -1, n - 1\) of the matrix. Is there a good choice of row operations to produce this determinant \(G_n\)?

21 \(S_1 = 3, S_2 = 8, S_3 = 21\). The rule looks like every second number in Fibonacci’s sequence \(3, 5, 8, 13, 21, 34, 55, \ldots\) so the guess is \(S_4 = 55\). Following the solution to Problem 30 with 3’s instead of 2’s on the diagonal confirms \(S_4 = 81 + 1 - 9 - 9 - 9 = 55\). Problem 32 directly proves \(S_n = F_{2n+2}\).

22 Changing 3 to 2 in the corner reduces the determinant \(F_{2n+2}\) by \(1\) times the cofactor of that corner entry. This cofactor is the determinant of \(S_{n-1}\) (one size smaller) which is \(F_{2n}\). Therefore changing 3 to 2 changes the determinant to \(F_{2n+2} - F_{2n}\), which is Fibonacci’s \(F_{2n+1}\).

23 (a) If we choose an entry from \(B\) we must choose an entry from the zero block; result zero. This leaves entries from \(A\) times entries from \(D\) leading to \((\det A)(\det D)\)

(b) and (c) Take \(A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\). See #25.

24 (a) All the lower triangular blocks \(L_k\) have 1’s on the diagonal and \(\det = 1.\) Then use \(A_k = L_kU_k\) to find \(\det U_k = \det A_k = 2, 6, -6\) for \(k = 1, 2, 3\)
(b) Equation (3) in this section gives the $k$th pivot as $\det A_k / \det A_{k-1}$. So $\det A_k = 5, 6, 7$ gives pivot $d_k = 5/1, 6/5, 7/6$.

25 Problem 23 gives $\det \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} = 1$ and $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = |A|$ times $|D - CA^{-1}B|$. By the product rule this is $|AD - ACA^{-1}B|$. If $AC = CA$ this is $|AD - CAA^{-1}B| = \det(AD - CB)$.

26 If $A$ is a row and $B$ is a column then $\det M = \det AB = \text{dot product of } A \text{ and } B$. If $A$ is a column and $B$ is a row then $AB$ has rank 1 and $\det M = \det AB = 0$ (unless $m = n = 1$). This block matrix $M$ is invertible when $AB$ is invertible which certainly requires $m \leq n$.

27 (a) $\det A = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$. Derivative with respect to $a_{11} = \text{cofactor } C_{11}$.

28 Row 1 – 2 row 2 + row 3 = 0 so this matrix is singular and $\det A$ is zero.

29 There are five nonzero products, all 1’s with a plus or minus sign. Here are the (row, column) numbers and the signs: $(1, 1)(2, 2)(3, 3)(4, 4) + (1, 2)(2, 1)(3, 4)(4, 3) - (1, 2)(2, 1)(3, 3)(4, 4) - (1, 1)(2, 2)(3, 4)(4, 3) - (1, 1)(2, 3)(3, 2)(4, 4)$. Total $-1$.

30 The 5 products in solution 29 change to $16 + 1 - 4 - 4 - 4$ since $A$ has 2’s and –1’s:

$$(2)(2)(2)(2) + (-1)(-1)(-1)(-1) - (-1)(-1)(2)(2) - (2)(2)(-1)(-1) - (2)(-1)(-1)(2) = 5 = n + 1.$$ 

31 $\det P = -1$ because the cofactor of $P_{14} = 1$ in row one has sign $(-1)^{1+4}$. The big formula for $\det P$ has only one term $(1 \cdot 1 \cdot 1 \cdot 1)$ with minus sign because three exchanges take 4, 1, 2, 3 into 1, 2, 3, 4; $\det(P^2) = (\det P)(\det P) = +1$ so

$$\det \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ is not right.}$$

32 The problem is to show that $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using Fibonacci’s rule:

$$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n} = 2F_{2n} + (F_{2n} - F_{2n-2}) = 3F_{2n} - F_{2n-2}.$$
33. The difference from 20 to 19 multiplies its 3 by 3 cofactor = 1: then det drops by 1.

34. (a) The last three rows must be dependent because only 2 columns are nonzero.

(b) In each of the 120 terms: Choices from the last 3 rows must use 3 different columns; at least one of those choices will be zero.

35. Subtracting 1 from the n, n entry subtracts its cofactor $C_{nn}$ from the determinant. That cofactor is $C_{nn} = 1$ (smaller Pascal matrix). Subtracting 1 from 1 leaves 0.

**Problem Set 5.3, page 283**

1. (a) $|A| = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3$, $|B_1| = \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = 6$, $|B_2| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ so $x_1 = -6/3 = -2$ and $x_2 = 3/3 = 1$.

(b) $|A| = 4$, $|B_1| = 3$, $|B_2| = 2$, $|B_3| = 1$.

Therefore $x_1 = 3/4$ and $x_2 = -1/2$ and $x_3 = 1/4$.

2. (a) $y = \begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix} = a$.

(b) $y = \det B_2 / \det A = (fg - id) / D$.

That is because $B_2$ with (1, 0, 0) in column 2 has $\det B_2 = fg - id$.

3. (a) $x_1 = 3/0$ and $x_2 = -2/0$: no solution

(b) $x_1 = x_2 = 0/0$: undetermined.

4. (a) $x_1 = \det([b \ a_2 \ a_3]) / \det A$, if $\det A \neq 0$. This is $|B_1| / |A|$.

(b) The determinant is linear in its first column so $|x_1 a_1 + x_2 a_2 + x_3 a_3 a_2 a_3|$ splits into $x_1 |a_1 a_2 a_3| + x_2 |a_2 a_2 a_3| + x_3 |a_3 a_2 a_3|$. The last two determinants are zero because of repeated columns, leaving $x_1 |a_1 a_2 a_3|$ which is $x_1 \det A$.

5. If the first column in $A$ is also the right side $b$ then $\det A = \det B_1$. Both $B_2$ and $B_3$ are singular since a column is repeated. Therefore $x_1 = |B_1| / |A| = 1$ and $x_2 = x_3 = 0$.

6. (a) $\begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

An invertible symmetric matrix has a symmetric inverse.
If all cofactors = 0 then $A^{-1}$ would be the zero matrix if it existed; cannot exist. (And also, the cofactor formula gives $\det A = 0$.) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has no zero cofactors but it is not invertible.

$$C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix} \quad \text{and } AC^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$ This is $(\det A)I$ and $\det A = 3$.

If we know the cofactors and $\det A = 1$, then $C^T = A^{-1}$ and also $\det A^{-1} = 1$.

Now $A$ is the inverse of $C^T$, so $A$ can be found from the cofactor matrix for $C$.

Take the determinant of $AC^T = (\det A)I$. The left side gives $\det AC^T = (\det A)(\det C)$ while the right side gives $(\det A)^n$. Divide by $\det A$ to reach $C = (\det A)^{n-1}$.

The cofactors of $A$ are integers. Division by $\det A = \pm 1$ gives integer entries in $A^{-1}$.

Both $\det A$ and $\det A^{-1}$ are integers since the matrices contain only integers. But $\det A^{-1} = 1/\det A$ so $\det A$ must be 1 or $-1$.

$$A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{has cofactor matrix } C = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -6 & 2 \\ 1 & 3 & -1 \end{bmatrix} \quad \text{and } A^{-1} = \frac{1}{5}C^T.$$ 

(a) Lower triangular $L$ has cofactors $C_{21} = C_{31} = C_{32} = 0$ \quad (b) $C_{12} = C_{21}$, $C_{31} = C_{13}, C_{32} = C_{23}$ make $S^{-1}$ symmetric. \quad (c) Orthogonal $Q$ has cofactor matrix $C = (\det Q)(Q^{-1})^T = \pm Q$ also orthogonal. Note $\det Q = 1$ or $-1$.

For $n = 5$, $C$ contains 25 cofactors and each 4 by 4 cofactor has 24 terms. Each term needs 3 multiplications: total 1800 multiplications vs.125 for Gauss-Jordan.

16 (a) Area $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 10 \quad$ (b) and (c) Area $10/2 = 5$, these triangles are half of the parallelogram in (a).

17 Volume $= \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 20$. Area of faces $= \begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = -2i - 2j + 8k$ length of cross product $= \begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = 6\sqrt{2}$

18 (a) Area $\begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix} = 5 \quad$ (b) $5 + \text{new triangle area } \begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 5 + 7 = 12$.

19 $\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 4 = \begin{vmatrix} 7 & 3 \end{vmatrix}$ because the transpose has the same determinant. See #22.
The edges of the hypercube have length $\sqrt{1+1+1+1} = 2$. The volume $\det H$ is $2^4 = 16$. ($H/2$ has orthonormal columns. Then $\det(H/2) = 1$ leads again to $\det H = 16$ in 4 dimensions.)

The maximum volume $L_1L_2L_3L_4$ is reached when the edges are orthogonal in $\mathbb{R}^4$. With entries 1 and $-1$ all lengths are $\sqrt{4} = 2$. The maximum determinant is $2^4 = 16$, achieved in Problem 20. For a $3 \times 3$ matrix, $\det A = (\sqrt{3})^3$ can’t be achieved by $\pm 1$.

This question is still waiting for a solution! An 18.06 student showed me how to transform the parallelogram for $A$ to the parallelogram for $A^T$, without changing its area. (Edges slide along themselves, so no change in baselength or height or area.)

The box has height 4 and volume $\det [1, 0, 0, 1] = 4$. $i \times j = k$ and $(k \cdot w) = 4$.

The $n$-dimensional cube has $2^n$ corners, $n2^{n-1}$ edges and $2n \ (n - 1)$-dimensional faces. Coefficients from $(2 + x)^n$ in Worked Example 2.4A. Cube from $2I$ has volume $2^n$.

The pyramid has volume $\frac{1}{3}$. The 4-dimensional pyramid has volume $\frac{1}{24}$ (and $\frac{1}{n!}$ in $\mathbb{R}^n$).

$x = r \cos \theta, y = r \sin \theta$ give $J = r$. This is the $r$ in polar area $r \, dr \, d\theta$. The columns are orthogonal and their lengths are 1 and $r$.

for triple integrals inside spheres. Those integrals have $\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$. 

$J = \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \sin \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & \theta \end{vmatrix} = \rho^2 \sin \phi$. This Jacobian is needed
29 From \(x, y\) to \(r, \theta\):
\[
\begin{bmatrix}
\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\
\frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y}
\end{bmatrix} = \begin{bmatrix}
x/r & y/r \\
-x/r^2 & x/r^2
\end{bmatrix} = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta/r & (\cos \theta)/r
\end{bmatrix}
\]
\(= \frac{1}{r} = \frac{1}{\text{Jacobian in } 27}\) The surprise was that \(\frac{dr}{dx}\) and \(\frac{dr}{dy}\) are both \(\frac{r}{x}\).

30 The triangle with corners \((0, 0), (6, 0), (1, 4)\) has area \((6)(4)/2 = 12\). Rotated by \(\theta = 60^\circ\) the area is unchanged. The determinant of the rotation matrix is
\[
J = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} = \begin{bmatrix}
1/2 & -\sqrt{3}/2 \\
\sqrt{3}/2 & 1/2
\end{bmatrix} = 1.
\]

31 Base area \(||u \times v|| = 10\), height \(||w|| \cos \theta = 2\), volume \((10)(2) = 20\).

32 The volume of the box is \(\det \begin{bmatrix}
2 & 4 & 0 \\
1 & 3 & 0 \\
1 & 2 & 2
\end{bmatrix} = 20\), agreeing with Problem 31.

33 \(\begin{vmatrix}
u_1 & v_2 & v_3 \\
u_1 & v_2 & v_3 \\
1 & 2 & 2
\end{vmatrix} = u_1 \begin{vmatrix}v_2 & v_3 \\
2 & 1
\end{vmatrix} - u_2 \begin{vmatrix}v_1 & v_3 \\
1 & 2
\end{vmatrix} + u_3 \begin{vmatrix}v_1 & v_2 \\
1 & 2
\end{vmatrix} \). This is \(u \cdot (v \times w)\).

34 \((w \times u) \cdot v = (v \times w) \cdot u = (u \times v) \cdot w\): Even permutation of \((u, v, w)\) keeps the same determinant. Odd permutations like \((u \times v) \cdot w\) will reverse the sign.

35 \(S = (2, 1, -1)\), area \(||PQ \times PS|| = ||(-2, -2, -1)|| = \sqrt{2^2 + 2^2 + 1^2} = 3\). The other four corners of the box can be \((0, 0, 0), (0, 0, 2), (1, 2, 2), (1, 1, 0)\). The volume of the tilted box with edges along \(P, Q, R\) is \(|\det| = 1\).

36 If \((1, 1, 0), (1, 2, 1), (x, y, z)\) are in a plane the volume is \(\det \begin{bmatrix}
x & y & z \\
1 & 1 & 0 \\
1 & 2 & 1
\end{bmatrix} = x - y + z = 0\).

The “box” with those edges is flattened to zero height.

37 \(\det \begin{bmatrix}
x & y & z \\
2 & 3 & 1 \\
1 & 2 & 3
\end{bmatrix} = 7x - 5y + z\) will be zero when \((x, y, z)\) is a combination of \((2, 3, 1)\) and \((1, 2, 3)\). The plane containing those two vectors has equation \(7x - 5y + z = 0\).

Volume = zero because the 3 box edges out from \((0, 0, 0)\) lie in a plane.
Doubling each row multiplies the volume by \(2^n\). Then \(2 \det A = \det(2A)\) only if \(n = 1\).

\(AC^T = (\det A)I\) gives \((\det A)(\det C) = (\det A)^n\). Then \(\det A = (\det C)^{1/3}\) with \(n = 4\). With \(\det A^{-1} = 1/\det A\), construct \(A^{-1}\) using the cofactors. *Invert to find A.*

The cofactor formula adds 1 by 1 determinants (which are just entries) times their cofactors of size \(n - 1\). Jacobi discovered that this formula can be generalized. For \(n = 5\), Jacobi multiplied each 2 by 2 determinant from rows 1-2 (with columns \(a < b\)) times a 3 by 3 determinant from rows 3-5 (using the remaining columns \(c < d < e\)).

The key question is + or − sign (as for cofactors). The product is given a + sign when \(a, b, c, d, e\) is an even permutation of \(1, 2, 3, 4, 5\). This gives the correct determinant +1 for that permutation matrix. More than that, all other \(P\) that permute \(a, b\) and separately \(c, d, e\) will come out with the correct sign when the 2 by 2 determinant for columns \(a, b\) multiplies the 3 by 3 determinant for columns \(c, d, e\).

The Cauchy-Binet formula gives the determinant of a square matrix \(AB\) (and \(AA^T\) in particular) when the factors \(A, B\) are rectangular. For (2 by 3) times (3 by 2) there are 3 products of 2 by 2 determinants from \(A\) and \(B\) (printed in boldface):

\[
\begin{bmatrix}
  a & b & c \\
  d & e & f \\
\end{bmatrix}
\begin{bmatrix}
  g & j \\
  h & k \\
  i & \ell \\
\end{bmatrix}
= 
\begin{bmatrix}
  a & b & c \\
  d & e & f \\
\end{bmatrix}
\begin{bmatrix}
  g & j \\
  h & k \\
  i & \ell \\
\end{bmatrix}
\begin{bmatrix}
  g & j \\
  h & k \\
  i & \ell \\
\end{bmatrix}
\]

Check \(A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}\) \(B = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 7 \end{bmatrix}\) \(AB = \begin{bmatrix} 14 & 30 \\ 30 & 66 \end{bmatrix}\)

*Cauchy-Binet:* \((4 - 2)(4 - 2) + (7 - 3)(7 - 3) + (14 - 12)(14 - 12) = 24\)

*det of \(AB\):* \((14)(66) - (30)(30) = 24\)

A 5 by 5 tridiagonal matrix has cofactor \(C_{11} = 4\) by 4 tridiagonal matrix. Cofactor \(C_{12}\) has only one nonzero at the top of column 1. That nonzero multiplies its 3 by 3 cofactor which is tridiagonal. So \(\det A = a_{11}C_{11} + a_{12}C_{12}\) = tridiagonal determinants of sizes 4 and 3. The number \(F_n\) of nonzero terms in \(\det A\) follows Fibonacci’s rule \(F_n = F_{n-1} + F_{n-2}\).