Problem Set 8.1, page 443

1. (a) To prove that \( \cos nx \) is orthogonal to \( \cos kx \) when \( k \neq n \), use \( (\cos nx)(\cos kx) = \frac{1}{2} \cos (n+k)x + \frac{1}{2} \cos (n-k)x \). Integrate from \( x = 0 \) to \( x = \pi \). What is \( \int_0^\pi \cos^2 kx \, dx \)?

(b) Correction From 0 to \( \pi \), \( \cos x \) is not orthogonal to \( \sin 2x \) (the book wrongly proposed \( \int_0^\pi \cos x \sin x \, dx \), but this is zero). For orthogonality of all sines and cosines, the period has to be \( 2\pi \).

Solution (a) \[
\int_0^\pi (\cos nx)(\cos kx) \, dx = \frac{1}{2} \int_0^\pi \cos((n+k)x) \, dx + \frac{1}{2} \int_0^\pi \cos((n-k)x) \, dx = 0 + 0
\]

Solution (b) \[
\int_0^\pi (\cos x)(\sin 2x) \, dx = \frac{-2}{3} \cos^3 x \bigg|_0^\pi = \frac{4}{3} \neq 0.
\]

Non-orthogonality comes from \( \int_0^\pi \cos mx \sin nx \, dx \) when \( m - n \) is an odd number.

2. Suppose \( F(x) = x \) for \( 0 \leq x \leq \pi \). Draw graphs for \( -2\pi \leq x \leq 2\pi \) to show three extensions of \( F \): a \( 2\pi \)-periodic even function and a \( 2\pi \)-periodic odd function and a \( \pi \)-periodic function.

Solution

\[ -2\pi \quad 0 \quad 2\pi \]

\[ -2\pi \quad 0 \quad 2\pi \]

\[ -2\pi \quad 0 \quad 2\pi \]

3. Find the Fourier series on \( -\pi \leq x \leq \pi \) for

(a) \( f_1(x) = \sin^3 x \), an odd function (sine series, only two terms)

Solution (a) The fast way is to know the identity \( \sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x \). This must be the Fourier sine series! It has only two terms.

More slowly, use Euler’s great formula to produce complex exponentials:

\[
(\sin x)^3 = \left( \frac{e^{ix} - e^{-ix}}{2i} \right)^3 = \frac{e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix}}{8i^3} = -\frac{1}{4} \sin 3x + \frac{3}{4} \sin x.
\]

Or slowly compute the usual formulas \( \int \sin^3 x \sin x \, dx \) and \( \int \sin^3 x \sin 3x \, dx \).
(b) \( f_2(x) = |\sin x| \), an even function (cosine series)

**Solution (b)**

\[
a_0 = \frac{1}{\pi} \int_0^\pi |\sin x| \, dx = \frac{2}{\pi}
\]

\[
a_k = \frac{1}{2\pi} \int_0^\pi |\sin x| \cos kx \, dx = -\frac{1}{4\pi} \left[ \frac{\cos(k-1)x}{k-1} + \frac{\cos(k+1)x}{k+1} \right]_{x=0}^{x=\pi}
\]

\[= 0 \text{ (odd } k\text{)} \quad \text{or} \quad -\frac{1}{4\pi} \left[ \frac{-2}{k-1} + \frac{-2}{k+1} \right] = \frac{k}{\pi(k^2 - 1)} \quad \text{(even } k\text{)}
\]

(c) \( f_3(x) = x \) for \(-\pi \leq x \leq \pi\) (sine series with jump at \( x = \pi \))

**Solution (c)**

\[
b_k = \frac{1}{\pi} \int_{-\pi}^\pi x \sin kx \, dx = \left[ \frac{1}{\pi k^2} \sin kx - \frac{x}{\pi k} \cos kx \right]_{-\pi}^{\pi}
\]

\[= \frac{1}{\pi} (\cos k\pi + \cos(-k\pi)) = -\frac{2}{k}(-1)^k.
\]

4 Find the complex Fourier series \( e^x = \sum c_k e^{ikx} \) on the interval \(-\pi \leq x \leq \pi\).

The even part of a function is \( \frac{1}{2}(f(x) + f(-x)) \), so that \( f_{\text{even}}(x) = f_{\text{even}}(-x) \). Find the cosine series for \( f_{\text{even}} \) and the sine series for \( f_{\text{odd}} \).

**Notice the jump at \( x = \pi \).**

**Solution**

\[
c_k = \frac{1}{2\pi} \int_{-\pi}^\pi e^x e^{-ikx} \, dx = \frac{1}{2\pi} \int_{-\pi}^\pi e^{x(1-ik)} \, dx
\]

\[= \left[ \frac{1}{2\pi(1-ik)} e^{x(1-ik)} \right]_{-\pi}^{\pi} = \frac{e^{\pi(1-ik)} - e^{-\pi(1-ik)}}{2\pi(1-ik)}
\]

The even part of the function is: \( \frac{1}{2}(e^x + e^{-x}) \). The cosine coefficients are

\[
a_0 = \frac{1}{4\pi} \int_{-\pi}^\pi (e^x + e^{-x}) \, dx = \frac{1}{2\pi} (e^\pi - e^{-\pi})
\]

\[
a_k = \frac{1}{2\pi} \int_{-\pi}^\pi (e^x + e^{-x}) \cos kx \, dx = \frac{2k \cosh[\pi] \sin[k\pi] + 2 \cos[k\pi] \sinh[\pi]}{\pi + k^2 \pi}
\]

The odd part of the function is: \( \frac{1}{2}(e^x - e^{-x}) \). The sine series is:

\[
b_k = \frac{1}{2\pi} \int_{-\pi}^\pi (e^x - e^{-x}) \sin kx \, dx = \frac{2 \cosh[\pi] \sin[k\pi] - 2k \cos[k\pi] \sinh[\pi]}{\pi + k^2 \pi}
\]

5 From the energy formula (21), the square wave sine coefficients satisfy

\[
\pi(b_1^2 + b_2^2 + \cdots) = \int_{-\pi}^\pi |\text{SW}(x)|^2 \, dx = \int_{-\pi}^\pi 1 \, dx = 2\pi.
\]
Substitute the numbers $b_k$ from equation (8) to find that $\pi^2 = 8(1 + \frac{1}{9} + \frac{1}{25} + \cdots)$.

**Solution** The sine coefficients for the odd square wave are

$$b_k = \frac{4}{\pi} \left( \frac{1 - (-1)^k}{2k} \right) = \frac{4}{\pi} \left( \frac{1}{1}, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \ldots \right)$$

Energy identity gives

$$\pi^2 = 8 \sum_{k=1}^{\infty} \left( \frac{1 - (-1)^k}{2k} \right)^2 = 8 \left( 1 + \frac{1}{9} + \frac{1}{25} + \cdots \right)$$

6) If a square pulse is centered at $x = 0$ to give

$$f(x) = 1 \quad \text{for} \quad |x| < \frac{\pi}{2}, \quad f(x) = 0 \quad \text{for} \quad \frac{\pi}{2} < |x| < \pi,$$

draw its graph and find its Fourier coefficients $a_k$ and $b_k$.

**Solution**

$$a_0 = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} dx = \frac{1}{2}$$

$$a_k = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos kx \; dx = \frac{2}{k\pi} \sin \frac{k\pi}{2} = \sin \left( \frac{k\pi}{2} \right)$$

$$b_k = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \sin kx \; dx = 0$$

7) Plot the first three partial sums and the function $x(\pi - x)$:

$$x(\pi - x) = \frac{8}{\pi} \left( \frac{\sin x}{1} + \frac{\sin 3x}{27} + \frac{\sin 5x}{125} + \cdots \right), 0 < x < \pi.$$
Chapter 8. Fourier and Laplace Transforms

\[ a_0 = \frac{1}{2\pi} \int_0^{\pi} \sin x \, dx = \frac{1}{2\pi} \left[ -\cos x \right]_0^{\pi} = \frac{1}{\pi} \]

\[ a_k = \frac{1}{\pi} \int_0^{\pi} \sin x \cos kx \, dx = -\frac{1}{2\pi} \left[ \frac{\cos(1-k)x + \cos(1+k)x}{1-k} \right]_0^{\pi} = \]

\[ [k \text{ even}] \frac{1}{\pi} \left( \frac{1}{1-k} + \frac{1}{1+k} \right) = \frac{2}{\pi(1-k^2)} \quad \text{[and 0 for k odd]} \]

\[ b_k = \frac{1}{\pi} \int_0^{\pi} \sin x \sin kx \, dx \text{ gives } b_1 = \frac{1}{2} \text{ and other } b_k = 0. \]

9 Suppose \( G(x) \) has period \( 2L \) instead of \( 2\pi \). Then \( G(x + 2L) = G(x) \). Integrals go from \(-L\) to \(L\) or from 0 to \(2L\). The Fourier formulas change by a factor \( \pi/L \):

\[ \text{The coefficients in } G(x) = \sum_{-\infty}^{\infty} C_k e^{ik\pi x/L} \text{ are } C_k = \frac{1}{2L} \int_{-L}^{L} G(x) e^{-ik\pi x/L} \, dx. \]

Derive this formula for \( C_k \): Multiply the first equation for \( G(x) \) by \( e^{-ik\pi x/L} \) and integrate both sides. Why is the integral on the right side equal to \( 2LC_k \)?

**Solution**

Multiply \( G(x) = \sum_{-\infty}^{\infty} C_k e^{ik\pi x/L} \) by \( e^{-ik\pi x/L} \). Integrate.

\[ \int_{-L}^{L} G(x) e^{-ik\pi x/L} \, dx = \int_{-L}^{L} e^{-ik\pi x/L} \sum_{-\infty}^{\infty} C_k e^{ik\pi x/L} \, dx \]

\[ \int_{-L}^{L} G(x) e^{-ik\pi x/L} \, dx = C_k \int_{-L}^{L} dx = 2LC_k \text{ (orthogonality)} \]

\[ C_k = \frac{1}{2L} \int_{-L}^{L} G(x) e^{-ik\pi x/L} \, dx \]

10 For \( G_{\text{even}} \), use Problem 9 to find the cosine coefficient \( A_k \) from \( (C_k + C_{-k})/2 \):

\[ G_{\text{even}}(x) = \sum_{0}^{\infty} A_k \cos \frac{k\pi x}{L} \quad \text{has} \quad A_k = \frac{1}{L} \int_{0}^{L} G_{\text{even}}(x) \cos \frac{k\pi x}{L} \, dx. \]

\( G_{\text{even}} \) is \( \frac{1}{2}(G(x) + G(-x)) \). Exception for \( A_0 = C_0 \): Divide by \( 2L \) instead of \( L \).

**Solution**

The result comes directly from \( \frac{1}{2}(C_k + C_{-k}) \).

11 Problem 10 tells us that \( a_k = \frac{1}{2}(c_k + c_{-k}) \) on the usual interval from 0 to \( \pi \).

Find a similar formula for \( b_k \) from \( c_k \) and \( c_{-k} \). In the reverse direction, find the complex coefficient \( c_k \) in \( F(x) = \sum c_k e^{ikx} \) from the real coefficients \( a_k \) and \( b_k \).
8.1. Fourier Series

**Solution**  
**Solution and correction** We are comparing two ways to write a Fourier series:

\[
\sum_{-\infty}^{\infty} c_k e^{ikx} = a_0 + \sum_{1}^{\infty} a_k \cos kx + \sum_{1}^{\infty} b_k \sin kx
\]

Pick out the terms for \(k\) and \(-k\):

\[
c_k e^{ikx} + c_{-k} e^{-ikx} = a_k \cos kx + b_k \sin kx
\]

Use Euler’s formula to reach cosines/sines on both sides:

\[
\left( c_k + c_{-k} \right) \cos kx + i \left( c_k - c_{-k} \right) \sin kx = a_k \cos kx + b_k \sin kx
\]

This shows that \(a_k = c_k + c_{-k}\) (correction from text) and \(b_k = i(c_k - c_{-k})\).

Reverse Euler’s formula to reach complex exponentials on both sides:

\[
c_k e^{ikx} + c_{-k} e^{-ikx} = \frac{1}{2} a_k \left( e^{ikx} + e^{-ikx} \right) + \frac{1}{2i} b_k \left( e^{ikx} - e^{-ikx} \right)
\]

This shows that \(c_k = \frac{1}{2} a_k + \frac{1}{2} b_k\) and \(c_{-k} = \frac{1}{2} a_k - \frac{1}{2i} b_k\).

Real functions with real \(a\)’s and \(b\)’s lead to \(c_{-k} = c_k\) (complex conjugates).

12 Find the solution to Laplace’s equation with \(u_0 = \theta\) on the boundary. Why is this the imaginary part of \(2(z - z^2/2 + z^3/3 \cdots) = 2 \log(1 + z)\)? Confirm that on the unit circle \(z = e^{i\theta}\), the imaginary part of \(2 \log(1 + z)\) agrees with \(\theta\).

**Solution** The sine series of the odd function \(f(\theta) = \theta\) has coefficients \(b_n =\)

\[
\frac{2}{\pi} \int_{0}^{\pi} \theta \sin n\theta d\theta = \frac{2}{\pi} \left[ \frac{1}{n^2} \sin n\theta - \frac{\theta}{n} \cos n\theta \right]_{0}^{\pi} = -\frac{2 \cos n\pi}{n} = \frac{2}{n^2} \left[ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots \right]
\]

The solution to Laplace’s equation inside the circle has factors \(r^n\):

\[
u(r, \theta) = \sum b_n r^n \sin n\theta = 2r \sin \theta - \frac{2r^2}{2} \sin 2\theta + \frac{2r^3}{3} \sin 3\theta \cdots
\]

\[
= \text{Im} \left[ 2z - \frac{2z^2}{2} + \frac{2z^3}{3} \cdots \right] = \text{Im} \left[ 2 \log(1 + z) \right].
\]

13 If the boundary condition for Laplace’s equation is \(u_0 = 1\) for \(0 < \theta < \pi\) and \(u_0 = 0\) for \(-\pi < \theta < 0\), find the Fourier series solution \(u(r, \theta)\) inside the unit circle. What is \(u\) at the origin \(r = 0\)?

**Solution** This 0-1 step function \(u_0(\theta)\) equals \(\frac{1}{2} + \frac{1}{4}\) (square wave). Equation (8) of the text gives the Fourier sine series for the square wave:

0-1 Step Function \(u_0(\theta) = \frac{1}{2} + \frac{2}{\pi} \left[ \frac{\sin \theta}{1} + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots \right]\)

Then the solution to Laplace’s equation includes factors \(r^n\):

\[
u(r, \theta) = \frac{1}{2} + \frac{2}{\pi} \left[ \frac{r \sin \theta}{1} + \frac{r^3 \sin 3\theta}{3} + \frac{r^5 \sin 5\theta}{5} + \cdots \right] = \frac{1}{2} \text{ at } r = 0.
\]
14 With boundary values \( u_0(\theta) = 1 + \frac{1}{4}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \cdots \), what is the Fourier series solution to Laplace’s equation in the circle? Sum this geometric series.

Solution Inside the circle we see factors \( r^n \) (and \( 1 + x + x^2 + \cdots = 1/(1-x) \)):

\[
u(r, \theta) = 1 + \frac{1}{2}re^{i\theta} + \frac{1}{4}r^2e^{2i\theta} + \cdots = 1/\left( 1 - \frac{1}{2}re^{i\theta} \right).
\]

15 (a) Verify that the fraction in Poisson’s formula (30) satisfies Laplace’s equation.

Solution (a) We could verify Laplace’s equation in \( r, \theta \) coordinates or recognize that every term in the sum (29) solves that equation:

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.
\]

(b) Find the response \( u(r, \theta) \) to an impulse at \( x = 0, y = 1 \) (where \( \theta = \pm \pi \)).

Solution (b) When the source is at the point \( \theta = \pm \pi \), this replaces \( r \cos \theta \) by \( -r \cos \theta \) in equation (30). Then the response to a point source is infinite at \( r = 1, \theta = \pm \pi \):

\[
u(r, \theta) = \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 + 2r \cos \theta}.
\]

16 With complex exponentials in \( F(x) = \sum c_ke^{ikx} \), the energy identity (21) changes to

\[
\int_{-\pi}^{\pi} |F(x)|^2 \, dx = 2\pi \sum |c_k|^2.
\]

Derive this by integrating \( (\sum c_ke^{ikx})(\sum c_k e^{-ikx}) \).

Solution All products \( e^{ikx}e^{-ikx} \) integrate to zero except when \( n = k \):

\[
\int_{-\pi}^{\pi} (c_ke^{ikx})(c_k e^{-ikx}) \, dx = 2\pi c_k \overline{c_k} = 2\pi |c_k|^2.
\]

The total energy is the sum over all \( k \).

17 A centered square wave has \( F(x) = 1 \) for \( |x| \leq \pi/2 \).

(a) Find its energy \( \int |F(x)|^2 \, dx \) by direct integration

Solution (a) \[
\int_{-\pi/2}^{\pi/2} |F(x)|^2 \, dx = \int_{-\pi/2}^{\pi/2} dx = \pi.
\]

(b) Compute its Fourier coefficients \( c_k \) as specific numbers

Solution (b) \[
c_k = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-ikx} \, dx = \left[ \frac{1}{2\pi} \frac{e^{-ikx}}{-ik} \right]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi ik} (e^{ik\pi/2} - e^{-ik\pi/2}) = \frac{1}{\pi k} \sin \left( \frac{k\pi}{2} \right)
\]

(c) Find the sum in the energy identity (Problem 8).

Solution (c) \[
\sin \frac{k\pi}{2} = 1, 0, -1, 0 \text{ (repeated)} \text{ so } 2\pi \sum |c_k|^2 = \frac{2}{\pi} \left( \frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \cdots \right) = 1.
\]
8.1. Fourier Series

18 \( F(x) = 1 + (\cos x)/2 + \cdots + (\cos nx)/2^n + \cdots \) is analytic: infinitely smooth.

(a) If you take 10 derivatives, what is the Fourier series of \( d^{10}F/dx^{10} \)?

(b) Does that series still converge quickly? Compare \( n^{10} \) with \( 2^n \) for \( n = 2^{10} \).

Solution (a) 10 derivatives of \( \cos nx \) gives \(-n^{10} \cos nx:\)

\[
\frac{d^{10}F}{dx^{10}} = \frac{-1}{2} \cos x - \frac{2^{10}}{2^2} \cos 2x - \frac{3^{10}}{2^3} \cos 3x \cdots - \frac{n^{10}}{2^n} \cos nx - \cdots
\]

Solution (b) Yes, \( 2^n \) gets large much faster than \( n^{10} \) so the series easily converges.

At \( n = 2^{10} = 1024 \) we have \( 2^n = 2^{1024} \), much larger than \( n^{10} = 2^{100} \).

19 If \( f(x) = 1 \) for \( |x| \leq \pi/2 \) and \( f(x) = 0 \) for \( \pi/2 < |x| < \pi \), find its cosine coefficients.

Can you graph and compute the Gibbs overshoot at the jumps?

Solution \( a_0 = \text{average value} = \frac{1}{2} \)

\[
a_k = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos kx \, dx = \frac{1}{\pi} \left[ \frac{1}{k} \right] \sin kx \bigg|_{-\pi/2}^{\pi/2} = \frac{2}{\pi} \sin \frac{k\pi}{2}
\]

20 Find all the coefficients \( a_k \) and \( b_k \) for \( F, I, \) and \( D \) on the interval \( -\pi \leq x \leq \pi \):

\[
F(x) = \delta \left( x - \frac{\pi}{2} \right) \quad I(x) = \int_0^x \delta \left( x - \frac{\pi}{2} \right) \, dx \quad D(x) = \frac{d}{dx} \delta \left( x - \frac{\pi}{2} \right).
\]

Solution (a) Integrate \( \cos kx \) and \( \sin kx \) against \( \delta \left( x - \frac{\pi}{2} \right) \) to get

\[
a_0 = \frac{1}{2\pi} \quad a_k = \frac{1}{\pi} \cos \frac{k\pi}{2} \quad \text{and} \quad b_k = \frac{1}{\pi} \sin \frac{k\pi}{2}
\]

Solution (b) The integral \( I(x) \) is the unit step function \( H(x - \pi/2) \) with jump at \( x = \pi/2 \):

\[
a_0 = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 1 \, dx = \frac{1}{4}
\]

\[
a_k = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos kx \, dx = \frac{1}{\pi} \left[ \sin k\pi \right] \left( \sin \frac{k\pi}{2} \right) = \frac{1}{\pi} \sin \frac{k\pi}{2}
\]

\[
b_k = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \sin kx \, dx = -\frac{1}{\pi} \left[ \cos k\pi \right] \left( \cos \frac{k\pi}{2} \right)
\]

Solution (c) \( D(x) \) is the “doublet” = derivative of the delta function \( \delta \left( x - \frac{\pi}{2} \right) \). You must integrate by parts (and \( D(-\pi) = D(\pi) = 0 \) fortunately).

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} D(x) \cos kx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta \left( x - \frac{\pi}{2} \right) \left( k \sin kx \right) \, dx
\]

So \( a_k \) for \( D(x) \) is \( kb_k \) in part (b), and \( b_k \) for \( D(x) \) is \(-ka_k \) in part (b).
For the one-sided tall box function in Example 4, with \( F = 1/h \) for \( 0 \leq x \leq h \), what is its odd part \( \frac{1}{2}(F(x) - F(-x)) \)? I am surprised that the Fourier coefficients of this odd part disappear as \( h \) approaches zero and \( F(x) \) approaches \( \delta(x) \).

**Solution**
Every function has an even part and an odd part:

\[
F_{\text{even}}(x) = \frac{1}{2}(F(x) + F(-x)) \quad F_{\text{odd}}(x) = \frac{1}{2}(F(x) - F(-x)) \quad F = F_{\text{even}} + F_{\text{odd}}
\]

For the one-sided box function, those even and odd parts are

\[
F_{\text{even}}(x) = \frac{1}{2h} \text{ for } |x| \leq h \quad F_{\text{odd}}(x) = -\frac{1}{h} \text{ for } -h \leq x \leq 0, + \frac{1}{h} \text{ for } 0 < x \leq h.
\]

The Fourier coefficients of \( F_{\text{odd}} \) don’t really “disappear” as \( h \to 0 \), because the energy \( \int |F_{\text{odd}}|^2 \, dx \) is growing. But it is growing in the high frequencies and any particular coefficient \( c_k \) (at a fixed frequency \( k \)) approaches zero as \( h \to 0 \).

22 Find the series \( F(x) = \sum c_k e^{ikx} \) for \( F(x) = e^x \) on \( -\pi \leq x \leq \pi \). That function \( e^x \) looks smooth, but there must be a hidden jump to get coefficients \( c_k \) proportional to \( 1/k \). Where is the jump?

**Solution**
When \( e^x \) is made into a periodic function there is a jump (or a drop) at \( x = \pi \). The drop from \( e^{\pi} \) to \( e^{-\pi} \) starts the next \( 2\pi \)-interval. That drop shows up as a factor multiplying the \( 1/k \) decay that all jump functions show in their Fourier expansion:

\[
c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-ikx} \, dx = \frac{1}{2\pi} \left[ \frac{e^{(1-ik)x}}{1-ik} \right]_{x=-\pi}^{x=\pi} = \frac{1}{2\pi} \frac{e^{\pi} - e^{-\pi}}{1-ik}.
\]

23 (a) (Old particular solution) Solve \( Ay'' + By' + Cy = e^{ikx} \).
(b) (New particular solution) Solve \( Ay'' + By' + Cy = \sum c_k e^{ikx} \).

**Solution**
This problem shows directly the power of **linearity** to deal with complicated forcing functions as combinations of simple forcing functions \( e^{ikx} \):

\[
Ay'' + By' + Cy = e^{ikx} \quad \text{has} \quad y_p = \frac{1}{(ik)^2A + ikB + C} e^{ikx} = Y_k e^{ikx}
\]

\[
Ay'' + By' + Cy = \sum c_k e^{ikx} \quad \text{has} \quad y_p = \sum c_k Y_k e^{ikx}.
\]

**Problem Set 8.2, page 453**

1 Multiply the three matrices in equation (11) and compare with \( F \). In which six entries do you need to know that \( i^2 = -1 \)? This is \( (w_1)^2 = w_2 \). If \( M = N/2 \), why is \( (w_N)^M = -1 \)?

**Solution**

2 Why is row \( i \) of \( \overline{F} \) the same as row \( N - i \) of \( F \) (numbered from 0 to \( N - 1 \))?
8.2. The Fast Fourier Transform

3 From Problem 8, find the 4 by 4 permutation matrix \( P \) so that \( F = P^{-1}F \). Check that 
\( P^2 = I \) so that \( P = P^{-1} \). Then from \( FP = 4I \) show that \( F^2 = 4P \).

It is amazing that \( F^4 = 16P^2 = 16I \). Four transforms of any \( c \) bring back 16 \( c \).

For all \( N, F^2/N \) is a permutation matrix \( P \) and \( F^4 = N^2I \).

Solution

4 Invert the three factors in equation (11) to find a fast factorization of \( F^{-1} \).

5 \( F \) is symmetric. Transpose equation (11) to find a new Fast Fourier Transform.

Solution

6 All entries in the factorization of \( F_6 \) involve powers of \( w = \text{sixth root of 1} \):

\[
F_6 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_3 \\ F_3 \end{bmatrix} \begin{bmatrix} P \\ P \end{bmatrix}.
\]

Write down these factors with 1, \( w, w^2 \) in \( D \) and powers of \( w^2 \) in \( F_3 \). Multiply!

Solution

7 Put the vector \( c = (1, 0, 1, 0) \) through the three steps of the FFT to find \( y = Fc \). Do the same for \( c = (0, 1, 0, 1) \).

Solution

8 Compute \( y = F_8c \) by the three FFT steps for \( c = (1, 0, 1, 0, 1, 0, 1, 0) \). Repeat the computation for \( c = (0, 1, 0, 1, 0, 1, 0, 1) \).

Solution

9 If \( w = e^{2\pi i/64} \) then \( w^2 \) and \( \sqrt{w} \) are among the _____ and _____ roots of 1.

Solution

10 \( F \) is a symmetric matrix. Its eigenvalues aren’t real. How is this possible?

Solution

The three great symmetric tridiagonal matrices of applied mathematics are \( K, B, C \). The eigenvectors of \( K, B, \) and \( C \) are discrete sines, cosines, and exponentials. The eigenvector matrices give the DST, DCT, and DFT — discrete transforms for signal processing. Notice that diagonals of the circulant matrix \( C \) loop around to the far corners.

\[
K = \begin{bmatrix} 2 & -1 \\ -1 & 2 & -1 \\ & \ddots & \ddots \\ & -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 2 & -1 \\ & \ddots & \ddots \\ & -1 & 2 \end{bmatrix}
\]

\[
C = \begin{bmatrix} 2 & -1 & & -1 \\ & -1 & 2 & -1 \\ & & \ddots & \ddots \\ -1 & & & -1 \\ & -1 & 2 \end{bmatrix} \quad K_{11} = K_{NN} = 2 \quad B_{11} = B_{NN} = 1 \quad C_{1N} = C_{N1} = -1
\]
11 The eigenvectors of $K_N$ and $B_N$ are the discrete sines $s_1, \ldots, s_N$ and the discrete cosines $c_0, \ldots, c_{N-1}$. Notice the eigenvector $e_0 = (1, 1, \ldots, 1)$. Here are $s_k$ and $c_k$—these vectors are samples of $\sin kx$ and $\cos kx$ from 0 to $\pi$.

$\begin{bmatrix}
\sin \frac{\pi k}{N+1}, 
\sin \frac{2\pi k}{N+1}, 
\ldots, 
\sin \frac{N\pi k}{N+1}
\end{bmatrix}$ and $\begin{bmatrix}
\cos \frac{\pi k}{2N}, 
\cos \frac{3\pi k}{2N}, 
\ldots, 
\cos \frac{(2N-1)\pi k}{2N}
\end{bmatrix}$

For 2 by 2 matrices $K_2$ and $B_2$, verify that $s_1$, $s_2$ and $c_0$, $c_1$ are eigenvectors.

**Solution**

12 Show that $C_3$ has eigenvalues $\lambda = 0, 3, 3$ with eigenvectors $e_0 = (1, 1, 1)$, $e_1 = (1, w, w^2)$, $e_2 = (1, w^2, w^4)$. You may prefer the real eigenvectors $(1, 1, 1)$ and $(1, 0, -1)$ and $(1, -2, 1)$.

**Solution**

13 Multiply to see the eigenvectors $e_k$ and eigenvalues $\lambda_k$ of $C_N$. Simplify to $\lambda_k = 2 - 2 \cos(2\pi k/N)$. Explain why $C_N$ is only semidefinite. It is not positive definite.

$$Ce_k = \begin{bmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{bmatrix} \begin{bmatrix}
w_k \\
w_{2k} \\
w_{(N-1)k}
\end{bmatrix} = (2 - w^k - w^{-k}) \begin{bmatrix}
w_k \\
w_{2k} \\
w_{(N-1)k}
\end{bmatrix}.$$

**Solution**

14 The eigenvectors $e_k$ of $C$ are automatically perpendicular because $C$ is a matrix. (To tell the truth, $C$ has repeated eigenvalues as in Problem 12. There was a plane of eigenvectors for $\lambda = 3$ and we chose orthogonal $e_1$ and $e_2$ in that plane.)

**Solution**

15 Write the 2 eigenvalues for $K_2$ and the 3 eigenvalues for $B_3$. Always $K_N$ and $B_{N+1}$ have the same $N$ eigenvalues, with the extra eigenvalue for $B_{N+1}$. (This is because $K = A^T A$ and $B = AA^T$.)

**Solution**

**Problem Set 8.5, page 477**

1 When the driving function is $f(t) = \delta(t)$, the solution starting from rest is the impulse response. The impulse is $\delta(t)$, the response is $y(t)$. Transform this equation to find the transfer function $Y(s)$. Invert to find the impulse response $y(t)$.

$$y'' + y = \delta(t) \text{ with } y(0) = 0 \text{ and } y'(0) = 0$$

**Solution** Take the Laplace Transform of $y'' + y = \delta(t)$ with $y(0) = y'(0) = 0$:

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 1$$

$$Y(s)(s^2 + 1) = 1$$

$$Y(s) = \frac{1}{s^2 + 1}$$ is the transform of $y(t) = \sin t$. 
2 (Important) Find the first derivative and second derivative of \( f(t) = \sin t \) for \( t \geq 0 \).
Watch for a jump at \( t = 0 \) which produces a spike (delta function) in the derivative.

**Solution** The first derivative of \( \sin(t) \) is \( \cos(t) \), and the second derivative is \( -\sin(t) + \delta(t) \).

3 Find the Laplace transform of the unit box function \( b(t) = \{1 \text{ for } 0 \leq t < 1\} = H(t) - H(t - 1) \). The unit step function is \( H(t) \) in honor of Oliver Heaviside.

**Solution** The unit box function is \( f(t) = H(t) - H(t - 1) \)

The transform is \( F(s) = \frac{1}{s} - \frac{e^{-st}}{s} = \frac{1}{s}(1 - e^{-s}) \)

The same result comes from \( F(s) = \int_0^\infty f(t) e^{-st} dt = \int_0^\infty e^{-st} dt \).

4 If the Fourier transform of \( f(t) \) is defined by \( \hat{f}(k) = \int f(t)e^{-ikt} dt \) and \( f(t) = 0 \) for \( t < 0 \), what is the connection between \( \hat{f}(k) \) and the Laplace transform \( F(s) \)?

**Solution** The Fourier Transform is the Laplace Transform with \( s = ik \): \( \hat{f}(k) = F(ik) \).

5 What is the Laplace transform \( R(s) \) of the standard ramp function \( r(t) = t \)?
For \( t < 0 \) all functions are zero. The derivative of \( r(t) \) is the unit step \( H(t) \).
Then multiplying \( R(s) \) by \( s \) gives \( \frac{1}{s} \).

**Solution** The Laplace Transform \( R(s) \) of the Ramp Function \( r(t) = t \) is

\[
R(s) = \int_0^\infty te^{-st} dt = \left[ \frac{te^{-st}}{s} \right]_0^\infty - \int_0^\infty \frac{e^{-st}}{s} dt = 0 - \frac{e^{-st}}{s^2} \bigg|_0^\infty = \frac{1}{s^2}
\]

Multiplying \( R(s) \) by \( s \) gives the Laplace transform \( 1/s \) of the step function.

6 Find the Laplace transform \( F(s) \) of each \( f(t) \), and the poles of \( F(s) \):
(a) \( f = 1 + t \)  
(b) \( f = t \cos \omega t \)  
(c) \( f = \cos(\omega t - \theta) \)  
(d) \( f = \cos^2 t \)  
(e) \( f = e^{-2t} \cos t \)  
(f) \( f = te^{-t} \sin \omega t \)

**Solution (a)** The transform of \( f(t) = 1 + t \) has a **double pole** at \( s = 0 \) :

\[
F(s) = \int_0^\infty (1 + t)e^{-st} dt = \int_0^\infty e^{-st} dt + \int_0^\infty te^{-st} dt = \frac{1}{s} + \frac{1}{s^2} = \frac{1 + s}{s^2}
\]

**Solution (b)**

\[
f(t) = t \cos(\omega t) = t \left( \frac{e^{i\omega t} + e^{-i\omega t}}{2} \right) = \frac{te^{i\omega t}}{2} + \frac{te^{-i\omega t}}{2}
\]
transforms to

\[
F(s) = \int_0^\infty \frac{te^{i(\omega - s)t}}{2} dt + \int_0^\infty \frac{te^{-i(\omega - s)t}}{2} dt
\]

\[
= \left. -\frac{e^{-t(s-i\omega)}(st-it\omega+1)}{2(s-i\omega)^2} \right|_0^\infty + \left. -\frac{e^{-t(s+i\omega)}(st+it\omega+1)}{2(s+i\omega)^2} \right|_0^\infty
\]

\[
= \frac{1}{2(s-i\omega)^2} + \frac{1}{2(s+i\omega)^2} = \frac{(s-i\omega)^2+(s+i\omega)^2}{2(s-i\omega)^2(s+i\omega)^2} = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}
\]

Poles occur at \( s = i\omega \) and \( s = -i\omega \), the two exponents of \( f(t) \).
218

Chapter 8. Fourier and Laplace Transforms

Solution (c) \( f(t) = \cos(\omega t - \theta) = \cos \omega t \cos \theta + \sin \omega t \sin \theta \) transforms to

\[
F(s) = \frac{s}{s^2 + \omega^2} \cos \theta + \frac{\omega}{s^2 + \omega^2} \sin \theta
\]

Poles occur at \( s = \pm i\omega \).

Solution (d)

\[
f(t) = \cos^2(t) = \frac{1}{4} (e^{it} + e^{-it})^2 = \frac{1}{4} (e^{2it} + 2 + e^{-2it})
\]

\[
F(s) = \int_0^\infty \frac{1}{4} (e^{2it} + e^{-2it} + 2e^{-st}) dt
\]

\[
= -\frac{1}{4(s-2t)} + \frac{1}{4(s+2t)} + \frac{1}{2s} = \frac{2s}{4(s^2 + 4)} + \frac{1}{2s} = \frac{s^2 + 2}{s(s^2 + 4)}
\]

Poles occur at \( s = 0 \) and \( s = \pm 2i \). Another way is to write \( \cos^2 t = \frac{1 + \cos 2t}{2} \)

Solution (e)

\[
f(t) = e^{-2t} \cos t = \frac{1}{2} e^{(i-2)t} + \frac{1}{2} e^{-(i+2)t}
\]

\[
F(s) = \int_0^\infty \frac{1}{2} e^{(i-2)t} e^{-st} dt + \int_0^\infty \frac{1}{2} e^{-(i+2)t} e^{-st} dt
\]

\[
= \left. \frac{1}{2(-i + 2 + s) + \frac{1}{2(i + 2 + s)} = \frac{s + 2}{(s + 2)^2 + 1} \right)
\]

Poles occur at the exponents \( s = -2 \pm i \) in \( f(t) \).

Solution (f)

\[
f(t) = te^{-t} \sin \omega t = \frac{t}{2i} e^{(i\omega - 1)t} - \frac{t}{2i} e^{-(i\omega + 1)t}
\]

\[
F(s) = \int_0^\infty \left( \frac{t}{2i} e^{(i\omega - 1)t} - \frac{t}{2i} e^{-(i\omega + 1)t} \right) e^{-st} dt
\]

\[
= \left. \frac{t}{2i} e^{(i\omega - 1 - s)t} \right|_0^\infty - \left. \frac{t}{2i} e^{-(i\omega + 1 + s)t} \right|_0^\infty
\]

\[
= \frac{ie^{-t(s - i\omega + 1)}(1 + t(s - i\omega + 1))}{2(s - i\omega + 1)^2} - \frac{ie^{-t(s + i\omega + 1)(1 + t(s + i\omega + 1))}{2(s + i\omega + 1)^2} \big|_0^\infty
\]

Poles of \( F(s) \) occur at \( s = -1 \pm i\omega \), the exponents of \( f(t) \).

7 Find the Laplace transform \( s \) of \( f(t) = \) next integer above \( t \) and \( f(t) = t \delta(t) \).

A staircase \( f(t) = \) \( t \) \( H(t) + H(t-1) + H(t-2) + \cdots \) next integer above \( t \) is a sum of step functions. The transform is

\[
\frac{1}{s} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \cdots = \frac{1}{s} (1 + e^{-s} + e^{-2s} + \cdots) = \frac{1}{s} \left( \frac{1}{1 - e^{-s}} \right).
\]

The differentiation rule \( \mathcal{L}(t f(t)) = -F'(s) \) with \( f(t) = \delta(t) \) and \( F(s) = 1 \) gives

\[
\mathcal{L}(t \delta(t)) = -\frac{d}{ds}(1) = 0 \text{ (this is correct because } t \delta(t) \text{ is the zero function).}
8.5. The Laplace Transform

8 Inverse Laplace Transform: Find the function \( f(t) \) from its transform \( F(s) \):

\[
\begin{align*}
(a) & \quad \frac{1}{s - 2\pi i} \\
(b) & \quad \frac{s + 1}{s^2 + 1} \\
(c) & \quad \frac{1}{(s-1)(s-2)} \\
(d) & \quad \frac{1}{s^2 + 2s + 10} \\
(e) & \quad \frac{e^{-s}}{s-a} \\
(f) & \quad 2s
\end{align*}
\]

Solution (a) \( F(s) = \frac{1}{s - 2\pi i} \) is the transform of \( f(t) = e^{2\pi it} \).

Solution (b) \( F(s) = \frac{s}{s^2 + 1} + \frac{1}{s^2 + 1} \) is the transform of \( f(t) = \cos t + \sin t \).

Solution (c) \( F(s) = \frac{1}{(s-1)(s-2)} = \frac{1}{s-2} - \frac{1}{s-1} \) is the transform of \( f(t) = e^{2t} - e^t \).

Solution (d)

\[
\begin{align*}
F(s) &= \frac{1}{s^2 + 2s + 10} = \frac{1}{(s+1+3i)(s+1-3i)} \\
&= \frac{i}{6(s+(1+3i))} - \frac{i}{6(s+(1-3i))} \\
f(t) &= \frac{i}{6} e^{-(1+3i)t} - \frac{i}{6} e^{-(1-3i)t} \\
&= -\frac{e^{-t} \sin(3t)}{3}
\end{align*}
\]

Solution (e)

\[
F(s) = \frac{e^{-s}}{s-a}
\]

\( f(t) = e^{at} H(t-1) = \text{shift of } e^{at} \)

Solution (f)

\[
F(s) = 2s
\]

\( f(t) = 2 \frac{d\delta}{dt} \)

9 Solve \( y'' + y = 0 \) from \( y(0) \) and \( y'(0) \) by expressing \( Y(s) \) as a combination of \( s/(s^2 + 1) \) and \( 1/(s^2 + 1) \). Find the inverse transform \( y(t) \) from the table.

Solution

\[
\begin{align*}
y'' + y &= 0 \\
s^2Y(s) - sy(0) - y'(0) + Y(s) &= 0 \\
Y(s)(s^2+1) &= sy(0) + y'(0) \\
Y(s) &= y(0)\frac{s}{s^2+1} + y'(0)\frac{1}{s^2+1}
\end{align*}
\]

The inverse transform is \( y(t) = y(0) \cos(t) + y'(0) \sin(t) \).

10 Solve \( y'' + 3y' + 2y = \delta \) starting from \( y(0) = 0 \) and \( y'(0) = 1 \) by Laplace transform. Find the poles and partial fractions for \( Y(s) \) and invert to find \( y(t) \).

Solution The transform of \( \frac{d^2y}{dt^2} + 3 \frac{dy}{dt} + 2y = \delta(t) \) with \( y(0) = 0 \) and \( y'(0) = 1 \) is
Chapter 8. Fourier and Laplace Transforms

\[ s^2Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0) + 2Y(s) = 1 \]
\[ Y'(s)(s^2 + 3s + 2) - 1 = 1 \]
\[ Y(s) = \frac{2}{(s + 1)(s + 2)} \]
\[ Y(s) = \frac{2}{s + 1} - \frac{2}{s + 2} \]
\[ y(t) = 2e^{-t} - 2e^{-2t} \]

11 Solve these initial-value problems by Laplace transform:

(a) \( y' + y = e^{i\omega t}, y(0) = 8 \)  
(b) \( y'' - y = e^t, y(0) = 0, y'(0) = 0 \)  
(c) \( y' + y = e^{-t}, y(0) = 2 \)  
(d) \( y'' + y = 6t, y(0) = 0, y'(0) = 0 \)  
(e) \( y' - i\omega y = \delta(t), y(0) = 0 \)  
(f) \( my'' + cy' + ky = 0, y(0) = 1, y'(0) = 0 \)

**Solution (a)**

\[ y' + y = e^{i\omega t} \text{ with } y(0) = 8 \]
\[ sY(s) - 8 + Y(s) = \frac{1}{s - i\omega} + 8 \]
\[ Y(s)(s + 1) = \frac{1}{(s + 1)(s - i\omega)} + \frac{8}{s + 1} \]
\[ Y(s) = \frac{1}{1 + i\omega} \left( \frac{1}{s - i\omega} - \frac{1}{s + 1} \right) + \frac{8}{s + 1} \]

Particular + null \( y(t) = \frac{1}{1 + i\omega} (e^{i\omega t} - e^{-t}) + 8e^{-t} \)

**Solution (b)**

\[ y'' - y = e^t \text{ with } y(0) = 0 \text{ and } y'(0) = 0 \]
\[ s^2Y(s) - Y(s) = \frac{1}{s - 1} \]
\[ Y(s) = \frac{\frac{1}{4(s + 1)^2} - \frac{1}{4(s - 1)} + \frac{1}{2(s - 1)^2}}{4(s + 1)} \]
\[ y(t) = \frac{e^{-t}}{4} - \frac{e^t}{4} + \frac{te^t}{2} \]

**Solution (c)**

\[ y' + y = e^{-t} \text{ with } y(0) = 2 \]
\[ sY(s) - 2 + Y(s) = \frac{1}{s + 1} \]
\[ Y(s) = \frac{\frac{1}{(s + 1)^2} + \frac{2}{s + 1}}{} \]
\[ y(t) = te^{-t} + 2e^{-t} \]

**Solution (d)**
8.5. The Laplace Transform

\[ y'' + y = 6t \quad \text{with} \quad y(0) = y'(0) = 0 \]

\[ s^2Y(s) + Y(s) = \frac{6}{s^2} \]

\[ Y(s)(s^2 + 1) = \frac{6}{s^2} \]

\[ Y(s) = \frac{6}{s^2} - \frac{3i}{s + i} + \frac{3i}{s - i} \]

\[ y(t) = 6t - 3ie^{-it} + 3ie^{it} = 6t - 6\sin t \]

**Solution (e)**

\[ y' - i\omega y = \delta(t) \quad \text{with} \quad y(0) = 0 \]

\[ sY(s) - i\omega Y(s) = 1 \]

\[ Y(s) = \frac{1}{s - i\omega} \]

\[ y(t) = e^{i\omega t} \]

**Solution (f)**

\[ my'' + cy' + ky = 0 \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0 \]

\[ ms^2Y(s) - msy(0) + csY(s) - cy(0) + kY(s) = 0 \]

\[ Y(s)(ms^2 + cs + k) = ms + c \]

\[ Y(s) = \frac{ms + c}{ms^2 + cs + k} \quad \text{has the form} \quad \frac{a}{s - s_1} + \frac{b}{s - s_2} \]

We used this Mathematica command to find \( y(t) \)

\text{Simplify}\[\text{InverseLaplaceTransform}\left[\frac{(ms + c)}{(ms^2 + cs + k)}, s, t\right]\]

\[ y(t) = \frac{e\left(\frac{c + \sqrt{c^2 - 4km}}{2m}\right)t}{2\sqrt{c^2 - 4km}} \left( e^{-\frac{c\sqrt{c^2 - 4km}}{2m}t} + e^{\frac{c\sqrt{c^2 - 4km}}{2m}t} \right) \]

12 The transform of \( e^{At} \) is \((sI - A)^{-1}\). Compute that matrix (the transfer function) when \( A = [1 \; 1; 1 \; 1] \). Compare the poles of the transform to the eigenvalues of \( A \).

**Solution**

When \( A = [1 \; 1; 1 \; 1] \) we have:

\[ (sI - A)^{-1} = \frac{1}{s^2 - 2s} \left[ \begin{array}{cc} s - 1 & 1 \\ -1 & s - 1 \end{array} \right] = \frac{1}{s^2 - 2s} \left[ \begin{array}{cc} s - 1 & 1 \\ 1 & s - 1 \end{array} \right]. \]

The poles of the system are \( s = 2 \) and \( s = 0 \), the eigenvalues of \( A \).

13 If \( dy/dt \) decays exponentially, show that \( sY(s) \to y(0) \) as \( s \to \infty \).

**Solution**

\[ sY(s) = \int_0^\infty se^{-st}y(t)\,dt \quad \text{(integrate by parts)} \]

\[ = \left[ e^{-st} \frac{dy}{dt} \right]_0^\infty - \int_0^\infty e^{-st} \frac{dy}{dt}\,dt \]

\[ = \int_0^\infty e^{-st} \frac{dy}{dt}\,dt + y(0) \to y(0) \quad \text{as} \quad s \to \infty \]

**Example:**

\[ \frac{dy}{dt} = e^{at} \quad \text{has} \quad sY(s) - y(0) = \frac{1}{s + a} \to 0 \quad \text{as} \quad s \to \infty \]
14 Transform Bessel’s time-varying equation $ty'' + y' + ty = 0$ using $\mathcal{L}[ty] = -dY/ds$ to find a first-order equation for $Y$. By separating variables or by substituting $Y(s) = C/\sqrt{1 + s^2}$, find the Laplace transform of the Bessel function $y = J_0$.

**Solution** The transform of $ty''$ applies the $\mathcal{L}(t, y'')$ rule to $y''$ instead of $y$:

$$\mathcal{L}(t, y'') = -\frac{d}{ds}(\text{transform of } y'') = -\frac{d}{ds}(s^2Y(s) - sy(0) - y'(0)).$$

Apply this to the transform of $td^2y/dt^2 + dy/dt + ty = 0$,

$$-2sY(s) - s^2\frac{dY}{ds} + y(0) + sY(s) - y(0) - \frac{dY}{ds} = 0$$

$$-sY(s) - s^2\frac{dY}{ds} - \frac{dY}{ds} = 0$$

$$sY(s) = -(s^2 + 1)\frac{dY}{ds}$$

$$\frac{dY}{Y(s)} = -\frac{s ds}{s^2 + 1}$$

$$\log Y(s) = \log \left(\frac{1}{\sqrt{s^2 + 1}}\right)$$

The transform of the Bessel solution $y = J_0$ is $Y(s) = \frac{1}{\sqrt{s^2 + 1}}$.

15 Find the Laplace transform of a single arch of $f(t) = \sin \pi t$.

**Solution** A single arch of $\sin \pi t$ extends from $t = 0$ to $t = 1$:

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^1 \sin(\pi t)e^{-st}dt = \int_0^1 \frac{e^{\pi t}-e^{-\pi t}}{2i}dt - \int_0^1 \frac{e^{-\pi t}+e^{\pi t}}{2i}dt$$

$$= \left[ \frac{e^{\pi t}-e^{-\pi t}}{2i(i\pi - s)} + \frac{e^{-\pi t}+e^{\pi t}}{2i(i\pi + s)} \right]_{t=0}^{t=1}$$

$$= \frac{e^{i\pi} - 1}{2i(i\pi - s)} + \frac{e^{-i\pi} - 1}{2i(i\pi + s)}$$

$$= \left( \frac{-e^{-s} - 1}{2i} \right) \left( \frac{1}{i\pi - s} - \frac{1}{i\pi + s} \right) = \left( \frac{e^{-s} + 1}{i} \right) \left( \frac{s}{\pi^2 + s^2} \right)$$

A faster and more direct approach: One arch of the sine curve agrees with $\sin \pi t$ + unit shift of $\sin \pi t$, because those cancel after one arch.

$\sin \pi t + \sin \pi (t - 1) = \sin \pi t + \sin \pi t \cos \pi = \sin \pi t - \sin \pi t = 0$.

16 Your acceleration $v' = c(v^* - v)$ depends on the velocity $v^*$ of the car ahead:

(a) Find the ratio of Laplace transforms $V^*(s)/V(s)$.

(b) If that car has $v^* = t$ find your velocity $v(t)$ starting from $v(0) = 0$.

**Solution** (a) Take the Laplace Transform of $\frac{dv}{dt} = c(v^* - v)$ assuming $v(0) = 0$;
8.5. The Laplace Transform

\[ sV(s) - v(0) = cV^*(s) - cV(s) \]
\[ V(s)(s + c) = cV^*(s) \]
\[ \frac{V^*(s)}{V(s)} = \frac{s + c}{c} \]

**Solution (b)** If \( v^*(t) = t \) then \( V^*(s) = \frac{1}{s^2} \). Therefore

\[ V(s)(s + c) = \frac{c}{s^2} \]
\[ V(s) = \frac{c}{s^3 + cs^2} \]
\[ = \frac{1}{c(s + c)} - \frac{1}{cs} + \frac{1}{s^2} \]
\[ v(t) = e^{-ct} - \frac{1}{c} + t \]

17 A line of cars has \( v'_n = c[v_{n-1}(t - T) - v_n(t - T)] \) with \( v_0(t) = \cos \omega t \) in front.

(a) Find the growth factor \( A = 1/(1 + i\omega e^{i\omega T}/c) \) in oscillation \( v_n = A^n e^{i\omega t} \).

(b) Show that \( |A| < 1 \) and the amplitudes are safely decreasing if \( cT < \frac{1}{2} \).

(c) If \( cT > \frac{1}{2} \) show that \( |A| > 1 \) (dangerous) for small \( \omega \). (Use \( \sin \theta < \theta \).)

Human reaction time is \( T \geq 1 \) sec and human aggressiveness is \( c = 0.4/\text{sec} \).

Danger is pretty close. Probably drivers adjust to be barely safe.

**Solution (a)** \( \frac{dv_n}{dt} = c(v_{n-1}(t - T) - v_n(t - T)) \) with \( v_n = A^n e^{i\omega t} \)

\[ i\omega A^n e^{i\omega t} = cA^{n-1} e^{i\omega(t-T)} - cA^n e^{i\omega(t-T)} \]
\[ A \frac{i\omega e^{i\omega T}}{c} = 1 - A \]
\[ A \left( 1 + \frac{i\omega e^{i\omega T}}{c} \right) = 1 \]

**Solution (b)**
For $|A| < 1$ we need $\left| 1 + \frac{i\omega}{c} e^{i\omega T} \right| > 1$

$$
\left| 1 - \frac{\omega}{c} \sin(\omega T) + \frac{\omega}{c} \cos(\omega T) \right| > 1
$$

$$
\left( 1 - \frac{\omega}{c} \sin(\omega T) \right)^2 + \frac{\omega^2}{c^2} \cos^2(\omega T) > 1
$$

$$
1 - \frac{2\omega}{c} \sin(\omega T) + \frac{\omega^2}{c^2} > 0
$$

Since $\sin(\omega T) < \omega T$, we are safe if $\frac{\omega^2}{c^2} > \frac{2\omega}{c} \omega T$ which is $cT < \frac{1}{2}$.

Solution (c) $\sin(\omega T) \approx \omega T$ when this number is small. Then the same steps show $|A| > 1$ when $cT > \frac{1}{2}$.

18 For $f(t) = \delta(t)$, the transform $F(s) = 1$ is the limit of transforms of tall thin box functions $b(t)$. The boxes have width $\epsilon \to 0$ and height $1/\epsilon$ and area 1.

Inside integrals, $b(t) = \begin{cases} 1/\epsilon & \text{for } 0 \leq t < \epsilon \\ 0 & \text{otherwise} \end{cases}$ approaches $\delta(t)$.

Find the transform $B(s)$, depending on $\epsilon$. Compute the limit of $B(s)$ as $\epsilon \to 0$.

Solution We begin by finding the transform of the box:

$$
B(s) = \int_0^\epsilon \frac{1}{\epsilon} e^{-st} dt = -\frac{1}{se} e^{-st} \bigg|_0^\epsilon = 1 - e^{-\epsilon s}
$$

We take the limit as $\epsilon \to 0$—the box approaches a delta function!

$$
B_\epsilon(s) = \lim_{\epsilon \to 0} \frac{1 - e^{-\epsilon s}}{se} = \lim_{\epsilon \to 0} 1 - \left( 1 - s \epsilon + \frac{1}{2} s^2 \epsilon^2 - \cdots \right) = 1.
$$

19 The transform $1/s$ of the unit step function $H(t)$ comes from the limit of the transforms of short steep ramp functions $r_\epsilon(t)$. These ramps have slope $1/\epsilon$:

Compute $R_\epsilon(s) = \int_0^\epsilon \frac{t}{\epsilon} e^{-st} dt + \int_\epsilon^\infty e^{-st} dt$. Let $\epsilon \to 0$.

Solution $R_\epsilon(s) = \int_0^\epsilon \frac{t}{\epsilon} e^{-st} dt + \int_\epsilon^\infty e^{-st} dt = \left[ \frac{e^{-st}(-st - 1)}{es^2} \right]_{t=0}^{t=\epsilon} + \left[ \frac{e^{-st}}{-s} \right]_{t=\epsilon}^{t=\infty}$

$$
= \frac{e^{-s\epsilon}(-s\epsilon - 1) + 1}{es^2} + \frac{e^{-s\epsilon}}{-s} = \frac{1 - e^{-s\epsilon}}{es^2}
$$

$$
\lim R_\epsilon(s) = \lim \frac{1 - (1 - s\epsilon + \frac{1}{2} s^2 \epsilon^2 - \cdots)}{es^2} = \frac{1}{s}.
$$
20 In Problems 18 and 19, show that the derivative of the ramp function $r_\epsilon(t)$ is the box function $b(t)$. The “generalized derivative” of a step is the _____ function.

Solution The generalized derivative of the short ramp $r_\epsilon(t)$ is the thin box $b(t)/\epsilon$. We say “generalized” because this is not a true derivative at $t = 0$: the ramp has zero slope left of $t = 0$ and nonzero slope right of $t = 0$. But the transforms of $r_\epsilon$ and $b_\epsilon$ follow the rule for derivatives.

The generalized derivative of a step function is a delta function.

21 What is the Laplace transform of $y'''(t)$ when you are given $Y(s)$ and $y(0), y'(0), y''(0)$?

Solution The Laplace Transform of $y'''(t)$ is $s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$.

22 The Pontryagin maximum principle says that the optimal control is “bang-bang”—it only takes on the extreme values permitted by the constraints. To go from rest at $x = 0$ to rest at $x = 1$ in minimum time, use maximum acceleration $A$ and deceleration $-B$. At what time $t$ do you change from the accelerator to the brake? (This is the fastest driving between two red lights.)

Solution The maximum principle requires full acceleration $A$ to an unknown time $t_0$ and then full deceleration $-B$ to reach $x = 1$ with zero velocity. The velocities are

$$v = At \quad \text{for} \quad t \leq t_0$$

$$v = At_0 - B(t - t_0) \quad \text{for} \quad t > t_0$$

Integrating the velocity $v = dx/dt$ gives the distance $x(t)$:

$$x = \frac{1}{2}At^2 \quad \text{for} \quad t < t_0$$

$$x = \frac{1}{2}At_0^2 \quad \text{at} \quad t = t_0$$

$$x = \frac{1}{2}At_0^2 + At_0(t - t_0) - \frac{1}{2}B(t - t_0)^2 \quad \text{for} \quad t > t_0$$

At the final time $T$ we reach $x = 1$ with velocity $v = 0$. This gives two equations for $t_0$ and $T$:

$$v = At_0 - B(T - t_0) = 0$$

$$x = At_0T - \frac{1}{2}At_0^2 - \frac{1}{2}B(T - t_0)^2 = 1$$

Substitute $T = \frac{1}{2}t_0(A + B)$ from the first equation into the second equation. This leaves an ordinary quadratic equation to solve for $t_0$.

Problem Set 8.6, page 453

1 Find the convolution $v \ast w$ and also the cyclic convolution $v \circledast w$:

(a) $v = (1, 2)$ and $w = (2, 1)$

Solution (a)

Convolution: $(1, 2) \ast (2, 1)$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Cyclic Convolution: $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$
226

Chapter 8. Fourier and Laplace Transforms

(b) \( v = (1, 2, 3) \) and \( w = (4, 5, 6) \).

Solution (b)

\[
(1, 2, 3) * (4, 5, 6) = \begin{bmatrix}
1 & 3 & 0 \\
2 & 1 & 0 \\
3 & 2 & 1 \\
0 & 3 & 2 \\
0 & 0 & 3
\end{bmatrix} \begin{bmatrix}
4 \\
5 \\
6
\end{bmatrix} = \begin{bmatrix}
4 \\
13 \\
28 \\
27 \\
18
\end{bmatrix}
\]

Cyclic Convolution:

\[
\begin{bmatrix}
1 & 3 & 2 \\
2 & 1 & 3 \\
3 & 2 & 1
\end{bmatrix} \begin{bmatrix}
4 \\
5 \\
6
\end{bmatrix} = \begin{bmatrix}
31 \\
31 \\
28
\end{bmatrix}
\]

2 Compute the convolution \( (1, 3, 1) * (2, 2, 3) = (a, b, c, d, e) \). To check your answer, add \( a + b + c + d + e \). That total should be 35 since \( 1 + 3 + 1 = 5 \) and \( 2 + 2 + 3 = 7 \) and \( 5 \times 7 = 35 \).

Solution

\[
\begin{bmatrix}
1 & 0 & 0 \\
3 & 1 & 0 \\
1 & 3 & 1 \\
0 & 1 & 3 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
2 \\
2 \\
3
\end{bmatrix} = \begin{bmatrix}
2 \\
8 \\
11 \\
11 \\
3
\end{bmatrix}
\]

\( 1 + 3 + 1 \) times \( 2 + 2 + 3 \) is \( 2 + 8 + 11 + 11 + 3 : (5) (7) = (35) \).

3 Multiply \( 1 + 3x + x^2 \) times \( 2 + 2x + 3x^2 \) to find \( a + bx + cx^2 + dx^3 + ex^4 \). Your multiplication was the same as the convolution \( (1, 3, 1) * (2, 2, 3) \) in Problem 8. When \( x = 1 \), your multiplication shows why \( 1 + 3 + 1 = 5 \) times \( 2 + 2 + 3 = 7 \) agrees with \( a + b + c + d + e = 35 \).

Solution

\[
(1 + 3x + x^2) \times (2 + 2x + 3x^2) = 2 + 2x + 3x^2 + 6x + 6x^2 + 9x^3 + 2x^2 + 2x^3 + 3x^4
\]

\[
= 2 + 8x + 11x^2 + 11x^3 + 3x^4
\]

At \( x = 1 \) this is again \( (5) \times (7) = (35) \).

4 (Deconvolution) Which vector \( v \) would you convolve with \( w = (1, 2, 3, 0) \) to get \( v \hat{\ast} w = (0, 1, 2, 3, 0) \)? Which \( v \) gives \( v \otimes w = (3, 1, 2) \)?

Solution

\[
\begin{bmatrix}
v_0 \\
v_1 \\
v_2 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
1 \\
2 \\
3 \\
3 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
2 \\
3 \\
2
\end{bmatrix}
\]

The first and last equation give \( v_0 = v_2 = 0 \). Substituting into the second, third, fourth equation gives \( v_1 = 1 \). Therefore \( v = (0, 1, 0, 0) \).

For cyclic convolution

\[
\begin{bmatrix}
1 & 3 & 2 \\
2 & 1 & 3 \\
3 & 2 & 1
\end{bmatrix} \begin{bmatrix}
v_0 \\
v_1 \\
v_2
\end{bmatrix} = \begin{bmatrix}
v_0 \\
v_1 \\
v_2
\end{bmatrix} \begin{bmatrix}
v_0 & v_2 & v_1 \\
v_1 & v_0 & v_2 \\
v_2 & v_1 & v_0
\end{bmatrix} \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} = \begin{bmatrix}
3 \\
1 \\
2
\end{bmatrix}
\]

This gives \( \begin{bmatrix}
v_0 \\
v_1 \\
v_2
\end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} \).
5 (a) For the periodic functions \( f(x) = 4 \) and \( g(x) = 2\cos x \), show that \( f \ast g \) is zero (the zero function)!

Solution (a) From equation (4) we have
\[
(f \ast g)(x) = \int_{0}^{2\pi} g(y)f(x-y) \, dy = 4 \int_{0}^{2\pi} 2\cos y \, dy = 4 \cdot 0 = 0 \quad \text{for all} \ x.
\]

(b) In frequency space \((k\text{-space})\) you are multiplying the Fourier coefficients of 4 and \( 2\cos x \). Those coefficients are \( c_0 = 4 \) and \( d_1 = d_{-1} = 1 \). Therefore every product \( c_k d_k \) is _____.

Solution (b) In frequency space \((k\text{-space})\) you are multiplying the Fourier coefficients of 4 and \( 2\cos x \). Those coefficients are \( c_0 = 4 \) and \( d_1 = d_{-1} = 1 \). Therefore every product \( c_k d_k \) is zero. These are the coefficients of the zero function.

6 For periodic functions \( f = \sum c_k e^{ikx} \) and \( g = \sum d_k e^{ikx} \), the Fourier coefficients of \( f \ast g \) are \( 2\pi c_k d_k \). Test this factor \( 2\pi \) when \( f(x) = 1 \) and \( g(x) = 1 \) by computing \( f \ast g \) from its definition (6.4).

Solution From equation (4):
\[
(f \ast g)(x) = \int_{0}^{2\pi} f(y)g(x-y) \, dy = \int_{0}^{2\pi} 1 \cdot 1 \, dy = 2\pi.
\]
The same convolution in \(k\text{-space}\) has \( c_0 = 1 \) and \( d_0 = 1 \) (all other \( c_k = d_k = 0 \)). Then \( 2\pi c_k d_k \) gives the correct coefficients \((2\pi \text{ and } 0)\) of the convolution \( f \ast g \) (which equals \( 2\pi \)).

7 Show by integration that the periodic convolution \( \int_{0}^{2\pi} \cos x \cos(t-x) \, dx \) is \( \pi \cos t \). In \( k\text{-space} \) you are squaring Fourier coefficients \( c_1 = c_{-1} = \frac{1}{2} \) to get \( \frac{1}{4} \) and \( \frac{1}{4} \); these are the coefficients of \( \frac{1}{2} \cos t \). The \( 2\pi \) in Problem 8 makes \( \pi \cos t \) correct.

Solution
\[
\int_{0}^{2\pi} \cos x \cos(t-x) \, dx = \int_{0}^{2\pi} \cos x(\cos t \cos x + \sin t \sin x) \, dx = \pi \cos t + 0.
\]

8 Explain why \( f \ast g \) is the same as \( g \ast f \) (periodic or infinite convolution).

Solution In Fourier space convolution \( f \ast g \) or \( f \circ g \) leads to multiplication \( c_k d_k \), which is certainly the same as \( d_k c_k \). So \( f \circ g = g \circ f \) in \( x\text{-space} \).

9 What 3 by 3 circulant matrix \( C \) produces cyclic convolution with the vector \( c = (1,2,3) \)? Then \( C \) equals \( c \oplus d \) for every vector \( d \). Compute \( c \oplus d \) for \( d = (0,1,0) \).

Solution The circulant matrix \( C = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \) gives cyclic convolution with \((1,2,3)\).

When \( d = (0,1,0) \) we have \( c \oplus d = C d = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \).
Chapter 8. Fourier and Laplace Transforms

10 What 2 by 2 circulant matrix $C$ produces cyclic convolution with $e = (1, 1)$? Show in four ways that this $C$ is not invertible. Deconvolution is impossible.

(1) Find the determinant of $C$. (2) Find the eigenvalues of $C$.
(3) Find $d$ so that $Cd = e \circledast d$ is zero. (4) $Fc$ has a zero component.

Solution The 2 by 2 circulant matrix $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ gives $(1,1) \circledast d = Cd$.

(1) The determinant of this matrix is zero.

(2) The eigenvalues of $C$ come from $\det \begin{bmatrix} 1 - \lambda & 1 \\ 1 & 1 - \lambda \end{bmatrix} = (1 - \lambda)^2 - 1 = 0$.
Then $(1 - \lambda)^2 = 1$ and $\lambda = 0, 2$. That zero eigenvalue means that the matrix $C$ is singular.

(3) $Cd = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so $C$ is not invertible: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in nullspace.

(4) The Fourier matrix $F$ gives $Fc = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. This again shows $\lambda = 2$ and 0.

11 (a) Change $b(x) \ast \delta(x - 1)$ to a multiplication $\hat{b}(k) \hat{\delta}(k)$:

The box $b(x) = \{1 \text{ for } 0 \leq x \leq 1\}$ transforms to $\hat{b}(k) = \int_0^1 e^{-ikx} \, dx$.

The shifted delta transforms to $\hat{\delta}(k) = \int \delta(x - 1)e^{-ikx} \, dx$.

(b) Show that your result $\hat{b} \hat{\delta}$ is the transform of a shifted box function. This shows how convolution with $\delta(x - 1)$ shifts the box.

Solution This question shows that continuous convolution with $\delta(x - 1)$ produces a shift in the box function $b(x)$, just like discrete convolution with the shifted delta vector $(\ldots, 0, 0, 1, \ldots)$ produces a one-step shift.

We compute $\delta(x - 1) \ast b(x)$ in $x$-space to find $b(x - 1)$, or in $k$-space to see the effect on the coefficients:

\[ \hat{b}(k) = \int_0^1 e^{-ikx} \, dx = \left[ \frac{e^{-ikx}}{-ik} \right]_{x=0}^{x=1} = \frac{1 - e^{-ik}}{ik} \]

Shifted box $e^{-ik} \left( \frac{1 - e^{-ik}}{ik} \right)$ agrees with $\int_1^2 e^{-ikx} \, dx = \left[ \frac{e^{-ikx}}{-ik} \right]_{x=1}^{x=2}$.

12 Take the Laplace transform of these equations to find the transfer function $G(s)$:

(a) $Ay'' + By' + Cy = \delta(t)$ \hspace{1cm} (b) $y' - 5y = \delta(t)$ \hspace{1cm} (c) $2y(t) - y(t - 1) = \delta(t)$

Solution (a) $A\bar{y}(s) + B\bar{y}'(s) + C\bar{y}(s) = 1$ gives the transfer function $\frac{1}{As^2 + Bs + C}$

Solution (b) $s\bar{y}(s) - 5\bar{y}(s) = 1$ gives the transfer function $\bar{y}(s) = \frac{1}{s - 5}$
8.6. Convolution (Fourier and Laplace) 229

Solution (c) $2Y(s) - Y(s)e^{-s} = 1$ gives the transfer function $Y(s) = \frac{1}{2 - e^{-s}}$

13 Take the Laplace transform of $y''' = \delta(t)$ to find $Y(s)$. From the Transform Table in Section 8.5 find $y(t)$. You will see $y'' = 1$ and $y''' = 0$. But $y(t) = 0$ for negative $t$, so your $y''$ is actually a unit step function and your $y'''$ is actually $\delta(t)$.

Solution $y''' = \delta$ transforms to $s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = 1$

Assume zero initial values to get $s^4Y(s) = 1$ and $Y(s) = \frac{1}{s^4}$ and $y^3 = \frac{t^3}{6}$.

This is also the solution to $y''' = 0$ with initial values $y, y', y'', y''' = 0, 0, 0, 1$.

14 Solve these equations by Laplace transform to find $Y(s)$. Invert that transform with the Table in Section 8.5 to recognize $y(t)$.

(a) $y' - 6y = e^{-t}, y(0) = 2$  
(b) $y'' + 9y = 1, y(0) = y'(0) = 0$.

Solution (a) The transform of $y' - 6y = e^{-t}$ with $y(0) = 2$ is

$$sY(s) - 2 - 6Y(s) = \frac{1}{s + 1}$$

$$Y(s) = \frac{2}{s - 6} + \frac{1}{(s + 1)(s - 6)}$$

$$= \frac{2}{s - 6} + \frac{1}{7(s - 6)} - \frac{1}{7(s + 1)}$$

$$= \frac{15}{7(s - 6)} - \frac{7}{7(s + 1)}$$

The inverse transform is $y(t) = \frac{15}{7}e^{6t} - \frac{7}{7}e^{-t}$

Solution (b) The transform of $y'' + 9y = 1$ with $y(0) = y'(0) = 0$ is

$$s^2Y(s) + 9Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s(s^2 + 9)}$$

$$= \frac{1}{9s} - \frac{1}{18(-3i + s)} - \frac{1}{18(3i + s)}$$

The inverse transform is $y(t) = \frac{1}{9} - \frac{1}{18}e^{3it} - \frac{1}{18}e^{-3it} = y_p + y_n$.

15 Find the Laplace transform of the shifted step $H(t - 3)$ that jumps from 0 to 1 at $t = 3$.

Solve $y' - ay = H(t - 3)$ with $y(0) = 0$ by finding the Laplace transform $Y(s)$ and then its inverse transform $y(t)$: one part for $t < 3$, second part for $t \geq 3$.

Solution The transform of $H(t - 3)$ multiplies $e^{-3s}$ by the transform $\frac{1}{s}$ of $H(t)$.

$$y' - ay = H(t - 3) \quad y(0) = 0$$

$$sY(s) - aY(s) = \frac{e^{-3s}}{s}$$

$$Y(s) = \frac{e^{-3s}}{s(s - 3)} = \frac{e^{-3s}}{3} \left( \frac{1}{s - 3} - \frac{1}{s} \right).$$

The inverse transform $y(t)$ is the shift of $\frac{1}{3}(e^{-3t} - 1)$; zero until $t = 3$. 


16 Solve \( y' = 1 \) with \( y(0) = 4 \)—a trivial question. Then solve this problem the slow way by finding \( Y(s) \) and inverting that transform.

Solution The trivial solution is: \( y = t + 4 \). The transform method gives
\[
sY(s) - 4 = \frac{1}{s} \\
Y(s) = \frac{1}{s^2} + \frac{4}{s} \\
y(t) = t + 4
\]

17 The solution \( y(t) \) is the convolution of the input \( f(t) \) with what function \( g(t) \)?

(a) \( y' - ay = f(t) \) with \( y(0) = 3 \)

Solution (a) \( y' - ay = f(t) \) with \( y(0) = 3 \)
\[
sY(s) - 3 - aY(s) = F(s) \\
Y(s) = \frac{3 + F(s)}{s - a} \\
y(t) = 3e^{-t} + f(t) * e^{-at}
\]

(b) \( y' - (\text{integral of } y) = f(t) \).

Solution (b) The transform of \( y' - (\text{integral of } y) = f(t) \) is \( sY(s) - \frac{Y(s)}{s} = F(s) \), if \( y(0) = 0 \).

The inverse transform of \( \frac{1}{s + \frac{a}{2}} = \frac{s}{s^2 - \frac{a^2}{4}} \) is \( \cos(at) \).

Then \( Y(s) = \frac{F(s)}{s - a} \) is the transform of the convolution \( f(t) * \cos(it) \).

18 For \( y' - ay = f(t) \) with \( y(0) = 3 \), we could replace that initial value by adding \( 3\delta(t) \) to the forcing function \( f(t) \). Explain that sentence.

Solution For a first order equation, an initial condition \( y(0) \) is equivalent to adding \( y(0)\delta(t) \) to the equation and starting that new equation at zero.

19 What is \( \delta(t) * \delta(t) \)? What is \( \delta(t - 1) * \delta(t - 2) \)? What is \( \delta(t - 1) \) times \( \delta(t - 2) \)?

Solution \( \delta(t) * \delta(t) = \delta(t) \)
\( \delta(t - 1) * \delta(t - 2) = \delta(t - 3) \)
\( \delta(t - 1) \) times \( \delta(t - 2) \) equals the zero function.

20 By Laplace transform, solve \( y' = y \) with \( y(0) = 1 \) to find a very familiar \( y(t) \).

Solution \( y' = y \) \( y(0) = 1 \)
\[
sY(s) - 1 = Y(s) \\
Y(s) = \frac{1}{s - 1} \text{ gives } y(t) = e^t.
\]
21 By Fourier transform as in (9), solve \(-y'' + y = \text{box function } b(x)\) on \(0 \leq x \leq 1\).

**Solution** The Fourier transform of \(-y'' + y = b(x)\) is

\[
(k^2 + 1) \hat{y}(k) = \hat{b}(k) = \int_{0}^{1} e^{-ikx} dx = \frac{1 - e^{-ik}}{ik}.
\]

This transform must be inverted to find \(y(x)\). In reality I would solve separately on \(x \leq 0\) and \(0 \leq x \leq 1\) and \(x \geq 1\). Then matching at the breakpoints \(x = 0\) and \(x = 1\) determines the free constants in the separate solutions.

22 There is a big difference in the solutions to \(y'' + By' + Cy = f(x)\), between the cases \(B^2 < 4C\) and \(B^2 > 4C\). Solve \(y'' + y = \delta\) and \(y'' - y = \delta\) with \(y(\pm \infty) = 0\).

**Solution** (a) The delta function produces a unit jump in \(y'\) at \(x = 0\):

\[
y'' + y = 0 \quad \text{has} \quad y = c_1 \cos x + c_2 \sin x \quad \text{for} \quad x < 0, \quad y = C_1 \sin x \quad \text{for} \quad x > 0.
\]

The jump in \(y'\) gives \(C_2 - c_2 = 1\). The condition on \(y(\pm \infty)\) does not apply to this first equation.

\[
y'' - y = 0 \quad \text{has} \quad y = ce^x \quad \text{for} \quad x < 0 \quad \text{and} \quad y = Ce^{-x} \quad \text{for} \quad x > 0; \quad \text{then} \quad y(\pm \infty) = 0.
\]

Matching \(y\) at \(x = 0\) gives \(c = C\).

Jump in \(y'\) at \(x = 0\) gives \(-C - c = 1\) so \(c = C = -\frac{1}{2}\).

Solution \(y(x) = -\frac{1}{2} e^x\) for \(x \leq 0\) and \(y(x) = -\frac{1}{2} e^{-x}\) for \(x \geq 0\)

23 *(Review)* Why do the constant \(f(t) = 1\) and the unit step \(H(t)\) have the same Laplace transform \(1/s\)? Answer: Because the transform does not notice .

**Solution** The Laplace Transform does not notice any values of \(f(t)\) for \(t < 0\).