DIFFERENTIAL EQUATIONS
AND
LINEAR ALGEBRA

MANUAL FOR INSTRUCTORS

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Problem Set 7.1, page 393

1 Suppose your pulse is measured at $b_1 = 70$ beats per minute, then $b_2 = 120$, then $b_3 = 80$. The least squares solution to three equations $v = b_1, v = b_2, v = b_3$ with $A^T = [1 1 1]$ is $\hat{v} = (A^T A)^{-1} A^T b = \boxed{\text{solution}}$. Use calculus and projections:

(a) Minimize $E = (v - 70)^2 + (v - 120)^2 + (v - 80)^2$ by solving $dE/dv = 0$.

Solution (a) $dE/dv = 2(v - 70) + 2(v - 120) + 2(v - 80) = 0$ at the minimizing $\hat{v}$.

Cancel the $2$’s: $3\hat{v} = 70 + 120 + 80 = 270$ so $\hat{v} = v_{\text{average}} = 90$

(b) Project $b = (70, 120, 80)$ onto $a = (1, 1, 1)$ to find $\hat{v} = a^T b/a^T a$.

Solution (b) The projection of $b$ onto the line through $a$ is $p = a\hat{v}$:

$$ b = \begin{bmatrix} 70 \\ 120 \\ 80 \end{bmatrix} \quad a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \hat{v} = \frac{a^T b}{a^T a} = \frac{270}{3} = 90. $$

2 Suppose $Av = b$ has $m$ equations $a_i^Tv = b_i$ in one unknown $v$. For the sum of squares $E = (a_1^Tv - b_1)^2 + \cdots + (a_m^Tv - b_m)^2$, find the minimizing $\hat{v}$ by calculus. Then form $A^T A\hat{v} = A^T b$ with one column in $A$, and reach the same $\hat{v}$.

Solution To minimize $E$ we solve $dE/dv = 0$. For $m = 3$ equations $a_i^Tv = b_i$,

$$ \frac{dE}{dv} = 2a_1(a_1^Tv - b_1) + 2a_2(a_2^Tv - b_2) + 2a_3(a_3^Tv - b_3) = 0 \quad \text{is zero when} \quad v = \frac{a^T b}{a^T a}. $$

When $A$ has one column, $A^T A\hat{v} = A^T b$ is the same as $(a^T a)\hat{v} = (a^T b)$.

3 With $b = (4, 1, 0, 1)$ at the points $x = (0, 1, 2, 3)$ set up and solve the normal equation for the coefficients $\hat{v} = (C, D)$ in the nearest line $C + Dx$. Start with the four equations $Av = b$ that would be solvable if the points fell on a line.

Solution The unsolvable equation has $m = 4$ points on a line: only $n = 2$ unknowns.

$$ Av = b \quad \text{is} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{leading to} \quad A^T A\hat{v} = A^T b : \quad \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 1/20 & 14/6 \\ -6/4 & -6 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/20 & -6 \end{bmatrix} \begin{bmatrix} 60 \\ -20 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} $$

The closest line to the four points is $b = 3 - x$.

4 In Problem 3, find the projection $p = Av$. Check that those four values lie on the line $C + Dx$. Compute the error $e = b - p$ and verify that $A^T e = 0$.

Solution The projection $p = A\hat{v}$ is
The best line \( C + Dx = 3 - x \) does produce \( p = (3,2,1,0) \) at the four points \( x = 0,1,2,3 \).

Multiply this \( e \) by \( A^T \) to get \( A^T e = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) as expected.

5 (Problem 3 by calculus) Write down \( E = ||b - Av||^2 \) as a sum of four squares: the last one is \((1 - \hat{C} - 3D)^2\). Find the derivative equations \( \partial E / \partial C = \partial E / \partial D = 0 \). Divide by 2 to obtain \( A^T A \hat{v} = A^T b \).

Solution Minimize \( E = (4 - C)^2 + (1 - C - D)^2 + (C - 2D)^2 + (1 - C - 3D)^2 \).

The partial derivatives are \( \partial E / \partial C = 0 \) and \( \partial E / \partial D = 0 \) at the minimum:

\[
-2(4 - C) - 2(1 - C - D) - 2(1 - C - 3D) = 0 \\
-2(1 - C - D) - 4(C - 2D) - 6(1 - C - 3D) = 0
\]

Factoring out \(-2\) and collecting terms this is the same equation \( A^T A \hat{v} = A^T b \):

\[
6 - 4C - 6D = 0 \quad \text{or} \quad \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.
\]

6 For the closest parabola \( C + Dt + Et^2 \) to the same four points, write down 4 unsolvable equations \( Av = b \) for \( v = (C, D, E) \). Set up the normal equations for \( \hat{v} \). If you fit the best cubic \( C + Dt + Et^2 + Ft^3 \) to those four points (thought experiment), what is the error vector \( e \)?

Solution The parabola \( C + Dt + Et^2 \) fits the 4 points exactly if \( Av = b \):

\[
t = 0 \quad C + 0D + 0E = 4 \\
t = 1 \quad C + 1D + 1E = 1 \\
t = 2 \quad C + 2D + 4E = 0 \\
t = 3 \quad C + 3D + 9E = 1
\]

\[
A^T A = \begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix} \quad \phi A^T b = \begin{bmatrix} 4 + 1 + 0 + 1 \\ 0 + 1 + 0 + 3 \\ 0 + 1 + 0 + 9 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \end{bmatrix}.
\]

The cubic \( C + Dt + Et^2 + Ft^3 \) can fit 4 points exactly, with error = zero vector.

7 Write down three equations for the line \( b = C + Dt \) to go through \( b = 7 \) at \( t = -1 \), \( b = 7 \) at \( t = 1 \), and \( b = 21 \) at \( t = 2 \). Find the least squares solution \( \hat{v} = (C, D) \) and draw the closest line.

Solution \( \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} \). The solution \( \hat{x} = [9, 4] \) comes from \( \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix} \).
9 Suppose the measurements at \( t = -1, 1, 2 \) are the errors 2, -6, 4 in Problem 8. Compute \( \hat{v} \) and the closest line to these new measurements. Explain the answer: \( b = (2, -6, 4) \) is perpendicular to ____ so the projection is \( p = 0 \).

**Solution** If \( b \) = previous error \( e \) then \( b \) is perpendicular to the column space of \( A \). Projection of \( b \) is \( p = 0 \).

10 Suppose the measurements at \( t = -1, 1, 2 \) are \( b = (5, 13, 17) \). Compute \( \hat{v} \) and the closest line \( e \). The error is \( e = 0 \) because this \( b \) is ____.

**Solution** If \( b = A\hat{x} = (5, 13, 17) \) then \( \hat{x} = (9, 4) \) and \( e = 0 \) since \( b \) is in the column space of \( A \).

11 Find the best line \( C + Dt \) to fit \( b = 4, 2, -1, 0, 0 \) at times \( t = -2, -1, 0, 1, 2 \).

**Solution** The least squares equation is \( \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix} \).

Solution: \( C = 1, D = -1 \). Line 1 \( t \). Symmetric \( t \)'s \( \Rightarrow \) diagonal \( A^T A \)

12 Find the plane that gives the best fit to the 4 values \( b = (0, 1, 3, 4) \) at the corners \((1,0)\) and \((0,1)\) and \((-1,0)\) and \((-1,-1)\) of a square. At those 4 points, the equations \( C + Dx + Ey = b \) are \( Av = b \) with 3 unknowns \( v = (C, D, E) \).

Solution \( \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \) has \( A^TA = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \) and \( A^Tb = -2 \).

The solution \( (C, D, E) = (2, -1, 2) \) gives the best plane \( 2 - x - \frac{2}{3}y \).

13 With \( b = 0, 8, 8, 20 \) at \( t = 0, 1, 3, 4 \) set up and solve the normal equations \( A^TAv = A^Tv \). For the best straight line \( C + Dt \), find its four heights \( p_t \) and four errors \( e_t \). What is the minimum value \( E = e_1^2 + e_2^2 + e_3^2 + e_4^2 \) ?

**Solution** \( A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \) and \( b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} \) give \( A^TA = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 26 \end{bmatrix} \) and \( A^Tb = \begin{bmatrix} 36 \\ 112 \end{bmatrix} \).

\( A^TA \hat{x} = A^Tb \) gives \( \hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \) and \( p = A\hat{x} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} \) and \( e = b - p = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix} \).

14 (By calculus) Write down \( E = \|b - Av\|^2 \) as a sum of four squares—the last one is \( (C + 4D - 20)^2 \). Find the derivative equations \( \partial E/\partial C = 0 \) and \( \partial E/\partial D = 0 \). Divide by 2 to obtain the normal equations \( A^TAv = A^Tb \).

**Solution** \( E = (C + 0D)^2 + (C + 1D - 8)^2 + (C + 3D - 8)^2 + (C + 4D - 20)^2 \). Then \( \partial E/\partial C = 2C + 2(C + D - 8) + 2(C + 3D - 8) + 2(C + 4D - 20) = 0 \) and \( \partial E/\partial D = 1 \cdot 2(C + D - 8) + 3 \cdot 2(C + 3D - 8) + 4 \cdot 2(C + 4D - 20) = 0 \).

These normal equations \( \partial E/\partial C = 0 \) and \( \partial E/\partial D = 0 \) are again \( \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix} \).

15 Which of the four subspaces contains the error vector \( e \)? Which contains \( p \)? Which contains \( \hat{v} \)?
7.1. Least Squares and Projections

Solution The error \( e \) is contained in the nullspace \( N(A^T) \), since \( A^T e = 0 \). The projection \( p \) is contained in the column space \( C(A) \). The vector \( \hat{v} \) of coefficients can be any vector in \( \mathbb{R}^n \).

16 Find the height \( C \) of the best horizontal line to fit \( b = (0, 8, 8, 20) \). An exact fit would solve the four unsolvable equations \( C = 0, C = 8, C = 8, C = 20 \). Find the 4 by 1 matrix \( A \) in these equations and solve \( A^T A \hat{v} = A^T b \).

Solution \( E = (C - 0)^2 + (C - 8)^2 + (C - 8)^2 + (C - 20)^2 \) and \( A^T = [1 \ 1 \ 1 \ 1] \).
\( A^T A = [4] \). \( A^T b = [36] \) and \( (A^T A)^{-1} A^T b = 9 = \text{best} C \). \( e = (-9, -1, -1, 11) \).

17 Write down three equations for the line \( b = C + Dt \) to go through \( b = 7 \) at \( t = -1, b = 7 \) at \( t = 1 \), and \( b = 21 \) at \( t = 2 \). Find the least squares solution \( \hat{v} = (C, D) \) and draw the closest line.

Solution \( \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 21 \end{bmatrix} \). The solution \( \hat{x} = \begin{bmatrix} 9 \\ 4 \end{bmatrix} \) comes from \( \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix} \).

18 Find the projection \( p = A \hat{v} \) in Problem 17. This gives the three heights of the closest line. Show that the error vector is \( e = (2, -6, 4) \). Why is \( Pe = 0 \) ?

Solution \( p = A \hat{x} = (5, 13, 17) \) gives the heights of the closest line. The error is \( b - p = (2, -6, 4) \). This error \( e \) has \( Pe = Pb - Pp = p - p = 0 \).

19 Suppose the measurements at \( t = -1, 1, 2 \) are the errors \( 2, -6, 4 \) in Problem 18. Compute \( \hat{v} \) and the closest line to these new measurements. Explain the answer: \( b = (2, -6, 4) \) is perpendicular to ______, so the projection is \( p = 0 \).

Solution If \( b = \text{error} e \) then \( b \) is perpendicular to the column space of \( A \). Projection \( p = 0 \).

20 Suppose the measurements at \( t = -1, 1, 2 \) are \( b = (5, 13, 17) \). Compute \( \hat{v} \) and the closest line and \( e \). The error is \( e = 0 \) because this \( b \) is ______ ?

Solution If \( b = A \hat{x} = (5, 13, 17) \) then \( \hat{x} = (9, 4) \) and \( e = 0 \) since \( b \) is in the column space of \( A \).

Questions 21–26 ask for projections onto lines. Also errors \( e = b - p \) and matrices \( P \).

21 Project the vector \( b \) onto the line through \( a \). Check that \( e \) is perpendicular to \( a \):

(a) \( b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) and \( a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) (b) \( b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \) and \( a = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix} \).

Solution (a) The projection \( p \) is
\[
p = a \frac{a^T b}{a^T a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{6}{3} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad e = b - p = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{perpendicular to} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
\]

Solution (b) In this case the projection is
\[
p = a \frac{a^T b}{a^T a} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix} \frac{-11}{11} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad \text{and} \quad e = b - p = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]
22 Draw the projection of \( b \) onto \( a \) and also compute it from \( p = \hat{v}a \):

(a) \( b = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \) and \( a = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

(b) \( b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( a = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

**Solution**

(a) The projection of \( b = (\cos \theta, \sin \theta) \) onto \( a = (1, 0) \) is \( p = (\cos \theta, 0) \)

(b) The projection of \( b = (1, 1) \) onto \( a = (1, -1) \) is \( p = (0, 0) \) since \( a^Tb = 0 \).

23 In Problem 22 find the projection matrix \( P = aa^T/a^Ta \) onto each vector \( a \). Verify in both cases that \( P^2 = P \). Multiply \( Pb \) in each case to find the projection \( p \).

**Solution**

\( P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and \( p = P_1b = \begin{bmatrix} \cos \theta \\ 0 \end{bmatrix} \), \( P_2 = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \) and \( p = P_2b = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).

24 Construct the projection matrices \( P_1 \) and \( P_2 \) onto the lines through the \( a \)'s in Problem 22. Is it true that \((P_1 + P_2)^2 = P_1 + P_2 \)? This would be true if \( P_1P_2 = 0 \).

**Solution**

The projection matrices \( P_1 \) and \( P_2 \) (note correction \( P_2 \) not \( P - 2 \)) are

\[ P_1 = \frac{aa^T}{a^Ta} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad P_2 = \frac{aa^T}{a^Ta} - \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} . \]

It is **not true** that \( (P_1 + P_2)^2 = P_1 + P_2 \). The sum of projection matrices is **not usually** a projection matrix.

25 Compute the projection matrices \( aa^T/a^Ta \) onto the lines through \( a_1 = (-1, 2, 2) \) and \( a_2 = (2, 2, -1) \). Multiply those two matrices \( P_1P_2 \) and explain the answer.

**Solution**

\( P_1 = \frac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} \), \( P_2 = \frac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} \).

\( P_1P_2 = \text{zero matrix because } a_1 \text{ is perpendicular to } a_2. \)

26 Continuing Problem 25, find the projection matrix \( P_3 \) onto \( a_3 = (2, -1, 2) \). Verify that \( P_1 + P_2 + P_3 = I \). The basis \( a_1, a_2, a_3 \) is orthogonal!

**Solution**

\( P_1 + P_2 + P_3 = \frac{1}{9} \left[ \begin{array}{ccc} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{array} \right] + \frac{1}{9} \left[ \begin{array}{ccc} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{array} \right] + \frac{1}{9} \left[ \begin{array}{ccc} 4 & -2 & -4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{array} \right] = I. \)

We can add projections onto **orthogonal vectors**. This is important.

27 Project the vector \( b = (1, 1) \) onto the lines through \( a_1 = (1, 0) \) and \( a_2 = (1, 2) \). Draw the projections \( p_1 \) and \( p_2 \) and add \( p_1 + p_2 \). The projections do not add to \( b \) because the \( a \)'s are not orthogonal.

**Solution**

The projections of \( (1, 1) \) onto the lines through \( (1, 0) \) and \( (1, 2) \) are \( p_1 = (1, 0) \) and \( p_2 = (3/5, 6/5) = (0.6, 1.2) \). Then \( p_1 + p_2 \neq b \).

28 (Quick and recommended) Suppose \( A \) is the \( 4 \times 4 \) identity matrix with its last column removed. \( A \) is \( 4 \times 3 \). Project \( b = (1, 2, 3, 4) \) onto the column space of \( A \). What shape is the projection matrix \( P \) and what is \( P? \)

**Solution**

\[ A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ P = \text{square matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ P = P \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}. \]
29 If \( A \) is doubled, then \( P = 2A(4A^TA)^{-1}2A^T \). This is the same as \( A(A^TA)^{-1}A^T \). The column space of \( 2A \) is the same as \( \boxed{.} \). Is \( \overrightarrow{v} \) the same for \( A \) and \( 2A \)?

**Solution** \( 2A \) has the same column space as \( A \). Same \( p \). But \( \overrightarrow{x} \) for \( 2A \) is \boxed{.} of \( \overrightarrow{x} \) for \( A \).

30 What linear combination of \((1, 2, -1)\) and \((1, 0, 1)\) is closest to \( b = (2, 1, 1) \)?

**Solution** \( \frac{1}{2}(1, 2, -1) + \frac{1}{2}(1, 0, 1) = (2, 1, 1) \). So \( b \) is in the plane: no error \( e \).

31 (Important) If \( P^2 = P \) show that \((I - P)^2 = I - P \). When \( P \) projects onto the column space of \( A \), \( I - P \) projects onto which fundamental subspace?

**Solution** If \( P^2 = P \) then \((I - P)(I - P) = I - PI - IP + P^2 = I - P \). When \( P \) projects onto the column space, \( I - P \) projects onto the left nullspace.

32 If \( P \) is the 3 by 3 projection matrix onto the line through \((1, 1, 1) \), then \( I - P \) is the projection matrix onto \boxed{.}

**Solution** \( I - P \) is the projection onto the plane \( x_1 + x_2 + x_3 = 0 \), perpendicular to the direction \((1, 1, 1) \):

\[
I - P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.
\]

33 Multiply the matrix \( P = A(A^TA)^{-1}A^T \) by itself. Cancel to prove that \( P^2 = P \).

**Solution** \( P(Pb) \) always equals \( Pb \): The vector \( Pb \) is in the column space so its projection is \boxed{.}

34 If \( A \) is square and invertible, the warning against splitting \((A^TA)^{-1} \) does not apply. Then \( AA^{-1}(A^TA)^{-1}A^T = I \) is true. When \( A \) is invertible, why is \( P = I \) and \( e = 0 \)?

**Solution** If \( A \) is invertible then its column space is all of \( \mathbb{R}^n \). So \( P = I \) and \( e = 0 \).

35 An important fact about \( A^TA \) is this: **If** \( A^TAx = 0 \) **then** \( Ax = 0 \).  **New proof**: The vector \( Ax \) is in the nullspace of \boxed{.} \( Ax \) is always in the column space of \boxed{.} To be in both of those perpendicular spaces, \( Ax \) must be zero.

**Solution** **If** \( A^TAx = 0 \) **then** \( Ax \) **is in the nullspace of** \( A^T \). **But** \( Ax \) **is always in the column space of** \( A \). **To be in both of those perpendicular spaces,** \( Ax \) **must be zero. So** \( A \) **and** \( A^T \) **have the same nullspace.**

**Notes on mean and variance and test grades**

If all grades on a test are 90, the mean is \( m = 90 \) and the variance is \( \sigma^2 = 0 \). Suppose the expected grades are \( g_1, \ldots, g_N \). Then \( \sigma^2 \) comes from **squaring distances to the mean**:

\[
\text{Mean} \quad m = \frac{g_1 + \cdots + g_N}{N} \quad \text{Variance} \quad \sigma^2 = \frac{(g_1 - m)^2 + \cdots + (g_N - m)^2}{N}
\]

After every test my class wants to know \( m \) and \( \sigma \). My expectations are usually way off.
36 Show that $\sigma^2$ also equals $\frac{1}{N}(g_1^2 + \cdots + g_N^2) - m^2$.

Solution Each term $(g_i - m)^2$ equals $g_i^2 - 2gm + m^2$, so

$$
\sigma^2 = \frac{(\text{sum of } g_i^2)}{N} - 2m\left(\frac{(\text{sum of } g_i)}{N}\right) + \frac{N^2 m^2}{N} = \frac{(\text{sum of } g_i^2)}{N} - \frac{N m N m + N m^2}{N}
$$

$$
= \frac{(\text{sum of } g_i^2)}{N} - m^2.
$$

37 If you flip a fair coin $N$ times (1 for heads, 0 for tails) what is the expected number $m$ of heads? What is the variance $\sigma^2$?

Solution For a fair coin you expect $N/2$ heads in $N$ flips. The variance $\sigma^2$ turns out to be $N/4$.

Problem Set 7.4, page 422

1 What solution to Laplace’s equation completes “degree 3” in the table of pairs of solutions? We have one solution $u = x^3 - 3xy^2$, and we need another solution.

Solution Start with $s = -y^3$. Then $s_{yy} = -6y$, and therefore we need $s_{xx} = 6y$. Integrating twice with respect to $x$ gives $3y^2x$. Therefore the second solution is $s(x, y) = -y^3 + 3x^2y$.

2 What are the two solutions of degree 4, the real and imaginary parts of $(x + iy)^4$? Check $u_{xx} + u_{yy} = 0$ for both solutions.

Solution Expanding $(x + iy)^4$ gives

$$(x + iy)^4 = x^4 - 6x^2y^2 + y^4 + (4x^3y - 4xy^3)i$$

Therefore the two solutions would be:

$$u(x, y) = x^4 - 6x^2y^2 + y^4 \text{ and } s(x, y) = 4x^3y - 4xy^3$$

Checking the first solution:

$$\frac{\partial^2}{\partial x^2}(x^4 - 6x^2y^2 + y^4) + \frac{\partial^2}{\partial y^2}(x^4 - 6x^2y^2 + y^4) = (12x^2 - 12y^2) + (-12x^2 + 12y^2) = 0$$

Checking the second solution:

$$\frac{\partial^2}{\partial x^2}(4x^3y - 4xy^3) + \frac{\partial^2}{\partial y^2}(4x^3y - 4xy^3) = (24xy - 0) + (0 - 24xy) = 0$$

3 What is the second $x$-derivative of $(x + iy)^n$? What is the second $y$-derivative? Those cancel in $u_{xx} + u_{yy}$ because $i^2 = -1$.

Solution The second $x$-derivative of $(x + iy)^n$ is:

$$\frac{\partial^2}{\partial x^2}(x + iy)^n = n(n-1)(x + iy)^{n-2}$$

The second $y$-derivative of $(x + iy)^n$ cancels that because

$$\frac{\partial^2}{\partial y^2}(x + iy)^n = i \cdot i \cdot n(n-1)(x + iy)^{n-2} = -n(n-1)(x + iy)^{n-2}$$
4 For the solved $2 \times 2$ example inside a $4 \times 4$ square grid, write the four equations (9) at the four interior nodes. Move the known boundary values 0 and 4 to the right hand sides of the equations. You should see $K^2D$ on the left side multiplying the correct solution $U = (U_{11}, U_{12}, U_{21}, U_{22}) = (1, 2, 2, 3)$.

Solution The equations at the interior node would be:

\[
\begin{align*}
4U_{11} - U_{21} - U_{01} - U_{12} - U_{10} &= 0 \\
4U_{12} - U_{22} - U_{02} - U_{13} - U_{11} &= 0 \\
4U_{21} - U_{31} - U_{11} - U_{22} - U_{20} &= 0 \\
4U_{22} - U_{32} - U_{12} - U_{23} - U_{21} &= 0
\end{align*}
\]

Substituting the known boundary values leaves:

\[
\begin{align*}
4U_{11} - U_{21} - U_{01} - U_{12} &= 4 \\
4U_{12} - U_{22} - U_{02} - U_{13} &= 8 \\
4U_{21} - U_{31} - U_{11} - U_{22} &= 0 \\
4U_{22} - U_{32} - U_{12} - U_{23} &= 4
\end{align*}
\]

Writing this in matrix form gives:

\[
\begin{bmatrix}
4 & -1 & 0 & -1 \\
-1 & 4 & -1 & 0 \\
0 & -1 & 4 & -1 \\
-1 & 0 & -1 & 4
\end{bmatrix}
\begin{bmatrix}
U_{11} \\
U_{12} \\
U_{21} \\
U_{22}
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
8 \\
0 \\
4
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
U_{11} \\
U_{12} \\
U_{21} \\
U_{22}
\end{bmatrix}
= 
\begin{bmatrix}
2 \\
3 \\
1 \\
2
\end{bmatrix}
\]

5 Suppose the boundary values on the $4 \times 4$ grid change to $U = 0$ on three sides and $U = 8$ on the fourth side. Find the four inside values so that each one is the average of its neighbors.

Solution The values at the 16 nodes will be:

\[
\begin{align*}
0 & & & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{3}{2} & \frac{3}{2} & 0 \\
0/4 & 4 & 4 & 0/4
\end{align*}
\]

Notice that the corner boundary values do not enter the 5-point equations around interior points. Every interior value must be the average of its four neighbors. By symmetry the two middle columns must be the same.

6 (MATLAB) Find the inverse $(K^2D)^{-1}$ of the $4 \times 4$ matrix displayed for the square grid.

Solution The circulant matrix $K^2D$ on page 422 has a circulant inverse:

\[
(K^2D)^{-1} = \frac{1}{24}
\begin{bmatrix}
7 & 2 & 1 & 2 \\
2 & 7 & 2 & 1 \\
1 & 2 & 7 & 2 \\
2 & 1 & 2 & 7
\end{bmatrix}
\]
7 Solve this Poisson finite difference equation (right side ≠ 0) for the inside values \( U_{11}, U_{12}, U_{21}, U_{22} \). All boundary values like \( U_{10} \) and \( U_{13} \) are zero. The boundary has \( i \) or \( j \) equal to 0 or 3, the interior has \( i \) and \( j \) equal to 1 or 2:

\[
4U_{ij} - U_{i-1,j} - U_{i+1,j} - U_{ij-1} - U_{ij+1} = 1
\]

at four inside points.

**Solution** The interior solution to the Poisson equation (on this small grid) is

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

On a larger grid \( U_{ij} \) will not be constant in the interior.

8 A 5 × 5 grid has a 3 by 3 interior grid: 9 unknown values \( U_{11} \) to \( U_{33} \). Create the 9 × 9 difference matrix \( K2D \).

**Solution** Order the points by rows to get \( U_{11}, U_{12}, U_{13}, U_{21}, U_{22}, U_{31}, U_{32}, U_{33} \). Then \( K2D \) is symmetric with 3 by 3 blocks:

\[
K2D = \begin{bmatrix}
A & -I & 0 \\
-I & A & -I \\
0 & -I & A
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
4 & -1 & 0 \\
-1 & 4 & -1 \\
0 & -1 & 4
\end{bmatrix}
\]

9 Use eig\((K2D)\) to find the nine eigenvalues of \( K2D \) in Problem 8. Those eigenvalues will be positive! The matrix \( K2D \) is symmetric positive definite.

**Solution** eig\((K2D)\) in Problem 8 produces 9 eigenvalues between 0 and 4:

The eigenvalues come from eig\((K2D)\) and explicitly from equation (11). Notice that pairs of eigenvalues add to 8. The eigenvalue distribution is symmetric around \( \lambda = 4 \):

\[
1.1716 \quad 2.5828 \quad 2.5828 \quad 4.0 \quad 4.0 \quad 4.0 \quad 5.4142 \quad 5.4142 \quad 6.8284
\]

10 If \( u(x) \) solves \( u_{xx} = 0 \) and \( v(y) \) solves \( v_{yy} = 0 \), verify that \( u(x)v(y) \) solves Laplace’s equation. Why is this only a 4-dimensional space of solutions? Separation of variables does not give all solutions—only the solutions with separable boundary conditions.

**Solution** If \( \frac{\partial^2 u}{\partial x^2} = 0 \) and \( \frac{\partial^2 v}{\partial y^2} = 0 \) then

\[
\frac{\partial^2 u(x)v(y)}{\partial x^2} + \frac{\partial^2 u(x)v(y)}{\partial y^2} = v(y) \frac{\partial^2 u(x)}{\partial x^2} + u(x) \frac{\partial^2 v(y)}{\partial y^2}
\]

\[
= v \cdot 0 + u \cdot 0 = 0
\]

Therefore \( u(x)v(y) \) solves Laplace’s equation. But the only solutions found this way are \( u(x)v(y) = (A + Bx)(C + Dy) \).
Problem Set 7.5, page 428

Problems 1 - 5 are about complete graphs. Every pair of nodes has an edge.

1 With \( n = 5 \) nodes and all edges, find the diagonal entries of \( A^T A \) (the degrees of the nodes). All the off-diagonal entries of \( A^T A \) are \(-1\). Show the reduced matrix \( R \) without row 5 and column 5. Node 5 is “grounded” and \( v_5 = 0 \).

Solution The complete graph (all edges included) has no zeros in \( A^T A \):

\[
A^T A = \begin{bmatrix}
4 & -1 & -1 & -1 & -1 \\
-1 & 4 & -1 & -1 & -1 \\
-1 & -1 & 4 & -1 & -1 \\
-1 & -1 & -1 & 4 & -1 \\
-1 & -1 & -1 & -1 & 4
\end{bmatrix}
\]

Singular! The grounded matrix would be

\[
(A^T A)_{\text{reduced}} = \begin{bmatrix}
4 & -1 & -1 & -1 \\
-1 & 4 & -1 & -1 \\
-1 & -1 & 4 & -1 \\
-1 & -1 & -1 & 4
\end{bmatrix}
\]

Invertible!

2 Show that the trace of \( A^T A \) (sum down the diagonal = sum of eigenvalues) is \( n^2 - n \). What is the trace of the reduced (and invertible) matrix \( R \) of size \( n - 1 \) ?

Solution \( A^T A \) is \( n \) by \( n \) and each diagonal entry is \( n - 1 \). Therefore the trace is \( n(n-1) = n^2 - n \). The reduced matrix \( R \) has \( n-1 \) diagonal entries, each still equal to \( n-1 \). Therefore the trace is \( (n-1)(n-1) = n^2 - 2n + 1 \).

3 For \( n = 4 \), write the 3 by 3 matrix \( R = (A_{\text{reduced}})^T (A_{\text{reduced}}) \). Show that \( RR^{-1} = I \) when \( R^{-1} \) has all entries \( \frac{1}{4} \) off the diagonal and \( \frac{2}{4} \) on the diagonal.

Solution Reduced matrix \( R = \begin{bmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3
\end{bmatrix} \)

\( R \) by its proposed inverse gives

\[
\begin{bmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3
\end{bmatrix}
\]

4 For every \( n \), the reduced matrix \( R \) of size \( n - 1 \) is invertible. Show that \( RR^{-1} = I \) when \( R^{-1} \) has all entries \( 1/n \) off the diagonal and \( 2/n \) on the diagonal.

Solution

\[
\frac{1}{n} \begin{bmatrix}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix} = \frac{1}{4} \begin{bmatrix}
6 - 1 - 1 & 3 - 2 - 1 & 3 - 1 - 2 \\
-2 + 3 - 1 & -1 + 6 - 1 & -1 + 3 - 2 \\
-2 - 1 + 3 & -1 - 2 + 3 & -1 - 1 + 6
\end{bmatrix} = I.
\]

5 Write the 6 by 3 matrix \( M = A_{\text{reduced}} \) when \( n = 4 \). The equation \( M v = b \) is to be solved by least squares. The vector \( b \) is like scores in 6 games between 4 teams (team 4 always scores zero; it is grounded). Knowing the inverse of \( R = M^T M \), what is the least squares ranking \( \hat{v}_1 \) for team 1 from solving \( M^T M \hat{v} = M^T b \)?

Solution Remove column 4 of \( A \) when node 4 is grounded \( (x_4 = 0) \).
Chapter 7. Applied Mathematics and $A^T A$

$$M = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

has independent columns

The least squares solution $\hat{v}$ to $Mv = b$ comes from $M^TM\hat{v} = M^Tb$. This $\hat{v}$ gives the predicted point spreads when all teams play all other teams. The first component $\hat{v}_1$ would come from the first row of $(M^TM)^{-1}$ multiplying by $M^Tb$. Note that $M^TM = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$ and $(M^TM)^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

6 For the tree graph with 4 nodes, $A^T A$ is in equation (1). What is the 3 by 3 matrix $R = (A^T A)_{\text{reduced}}$? How do we know it is positive definite?

Solution The reduced form of $A^T A$ removes row 4 and column 4:

Singular $A^T A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ reduces to invertible $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

The first is positive semidefinite ($A$ has dependent columns). The second is positive definite (the reduced $A$ has 3 independent columns).

7 (a) If you are given the matrix $A$, how could you reconstruct the graph?

Solution Each row of $A$ tells you an edge in the graph.

(b) If you are given $L = A^T A$, how could you reconstruct the graph (no arrows)?

Solution Each nonzero off the main diagonal of $A^T A$ tells you an edge.

(c) If you are given $K = A^T CA$, how could you reconstruct the weighted graph?

Solution Each nonzero off the main diagonal tells you the weight of that edge.

8 Find $K = A^T CA$ for a line of 3 resistors with conductances $c_1 = 1$, $c_2 = 4$, $c_3 = 9$. Write $K_{\text{reduced}}$ and show that this matrix is positive definite.

Solution A circle of three resistors has 3 edges and 3 nodes:

$$A^T CA = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -4 & -1 \\ -4 & 13 & -9 \\ -1 & -9 & 10 \end{bmatrix}$$

is only semidefinite

$$(A^T CA)_{\text{reduced}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 13 \end{bmatrix}$$

The determinant tests $5 > 0$ and $(5)(13) > 4^2$ are passed.
9 A 3 by 3 square grid has \( n = 9 \) nodes and \( m = 12 \) edges. Number nodes by rows.

(a) How many nonzeros among the 81 entries of \( L = A^T A \)?

**Solution** The 9 nodes ordered by rows have 2, 3, 2, 3, 4, 3, 2, 3, 2 neighbors around them. Those add to 24 nonzeros off the diagonal. The 9 diagonal entries make 33 nonzeros out of \( 9^2 = 81 \) entries in \( L = A^T A \).

(b) Write down the 9 diagonal entries in the degree matrix \( D \): they are not all 4.

**Solution** Those 9 numbers are the degrees of the 9 nodes (= diagonal entries in \( A^T A \)).

(c) Why does the middle row of \( L = D - W \) have four -1’s? Notice \( L = K^2 D \)!

**Solution** The middle node in the grid has 4 neighbors.

10 Suppose all conductances in equation (5) are equal to \( c \). Solve equation (6) for the voltages \( v_2 \) and \( v_3 \) and find the current \( I \) flowing out of node 1 (and into the ground at node 4). What is the “system conductance” \( I/V \) from node 1 to node 4?

This overall conductance \( I/V \) should be larger than the individual conductances \( c \).

**Solution** The reduced equation (6) with conductances \( = c \) is

\[
\begin{bmatrix}
3c & -c \\
-c & 2c
\end{bmatrix}
\begin{bmatrix}
v_2 \\
v_3
\end{bmatrix} =
\begin{bmatrix}
cV \\
cV
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
v_2 \\
v_3
\end{bmatrix} =
\begin{bmatrix}
0.6V \\
0.8V
\end{bmatrix}.
\]

Then the flows on the five edges in Figure 7.6 use \( A \) in equation (2). Remember the minus sign:

\[
-cAv = -c
\begin{bmatrix}
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
0 & -1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
V \\
\frac{1}{6}V \\
\frac{1}{8}V \\
0
\end{bmatrix} =
\begin{bmatrix}
0.4 \\
0.2 \\
-0.2 \\
1.0 \\
0.6
\end{bmatrix}.
\]

The total flow (on edges 1+2+4 out of node 1, or on edges 3+4 into the grounded node 4, is \( I = 1.6eV \). The overall system conductance is 1.6\( c \), greater than the individual conductance \( c \) on each edge.

11 The multiplication \( A^T A \) can be columns of \( A^T \) times rows of \( A \). For the tree with \( m = 3 \) edges and \( n = 4 \) nodes, each (column times row) is \( 4 \times 1 \times 1 \times 4 = 4 \times 4 \). Write down those three column-times-row matrices and add to get \( L = A^T A \).

**Solution** Suppose the 3 tree edges go out of node 1 to nodes 2, 3, 4. (The problem allows to choose other trees, including a line of 4 nodes.) Then

\[
A =
\begin{bmatrix}
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1
\end{bmatrix} \quad A^T A =
\begin{bmatrix}
3 & -1 & -1 \\
-1 & 1 & 0 \\
-1 & 0 & 1 \\
-1 & 0 & 0 & 1
\end{bmatrix} = \text{sum of (columns of} A^T \text{)( rows of } A) \]

\[
= \begin{bmatrix}
-1 \\
1 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1
\end{bmatrix}
+ \begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]
12 A graph with two separate 3-node trees is not connected. Write its 6 by 4 incidence matrix $A$. Find two solutions to $Av = 0$, not just one solution $v = (1, 1, 1, 1, 1, 1)$. To reduce $A^T A$ we must ground two nodes and remove two rows and columns.

**Solution** The incidence matrix for two 3-node trees is

$$A = \begin{bmatrix} A_{\text{tree}} & 0 \\ 0 & A_{\text{tree}} \end{bmatrix}$$

with $A_{\text{tree}} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (for example)

The columns of $A_{\text{tree}}$ add to zero so we have 2 independent solutions to $Av = 0$:

$$v = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

from $A_{\text{tree}}$.

13 “Element matrices” from column times row appear in the finite element method. Include the numbers $c_1, c_2, c_3$ in the element matrices $K_1, K_1, K_3$.

$$K_i = \text{row } i \text{ of } A^T \quad (c_i) \quad \text{row } i \text{ of } A \quad K = A^T CA = K_1 + K_2 + K_3.$$

Write the element matrices that add to $A^T A$ in (1) for the 4-node line graph.

$$A^T A = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \\ 0 & 0 \\ K_3 \\ 0 \end{bmatrix}$$

assembly of the nonzero entries of $K_1 + K_2 + K_3$

from edges 1, 2, and 3

**Solution** The three “element matrices” for the three edges come from multiplying the three columns of $A^T$ by the three rows of $A$. Then $A^T A$ equals

$$= \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

When the diagonal matrix $C$ is included, those are multiplied by $c_1, c_2$, and $c_3$. Those products produce 2 by 2 blocks of nonzeros in $4 \times 4$ matrices:

$$K_1 = c_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad K_2 = c_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad K_3 = c_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Then $A^T CA = K_1 + K_2 + K_3$. This “assembly” of the element stiffness matrices just requires placing the nonzeros correctly into the final matrix $A^T CA$.

14 An $n$ by $n$ grid has $n^2$ nodes. How many edges in this graph? How many interior nodes? How many nonzeros in $A$ and in $L = A^T A$? There are no zeros in $L^{-1}$!

**Solution** An $n$ by $n$ grid has $n$ horizontal rows ($n-1$ edges on each row) and $n$ vertical columns ($n-1$ edges down each column). Altogether $2n(n-1)$ edges. There are
\[(n - 2)^2\] interior nodes—a square grid with the boundary nodes removed to reduce \(n\) to \(n - 2\).

Every edge produces 2 nonzeros \((-1\) and \(+1\)) in \(A\). Then \(A\) has \(4n(n - 1)\) nonzeros.

The matrix \(A^T A\) has size \(n^2\) with \(n^2\) diagonal nonzeros—and off the diagonal of \(A^T A\) there are two \(-1\)'s for each edge: altogether \(n^2 + 4n(n - 1) = 5n^2 - 4n\) nonzeros out of \(n^3\) entries. For \(n = 2\), this means 12 nonzeros in a 4 by 4 matrix.

15 When only \(e = C^{-1} w\) is eliminated from the 3-step framework, equation (??) shows

\[
\text{Saddle-point matrix} \quad \begin{bmatrix} C^{-1} & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} b \\ f \end{bmatrix}.
\]

Multiply the first block row by \(A^T C\) and subtract from the second block row:

\[
\text{After block elimination} \quad \begin{bmatrix} C^{-1} & A \\ 0 & -A^T C A \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} b \\ f - A^T C b \end{bmatrix}.
\]

After \(m\) positive pivots from \(C^{-1}\), why does this matrix have negative pivots? The two-field problem for \(w\) and \(v\) is finding a saddle point, not a minimum.

**Solution** The three equations \(e = b - Av\) and \(w = Ce\) and \(A^T w = f\) reduce to two equations when \(e\) is replaced by \(C^{-1} w\):

\[
\begin{align*}
C^{-1} w &= b - Av \\
A^T w &= f
\end{align*}
\]

become

\[
\begin{bmatrix} C^{-1} & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} b \\ f \end{bmatrix}.
\]

Multiply the first equation by \(A^T C\) to get \(A^T w = A^T C b - A^T C A v\). Subtract from the second equation \(A^T w = f\), to eliminate \(w\):

\[
A^T C b - A^T C A v = f.
\]

This gives the second row of the block matrix after elimination:

\[
\begin{bmatrix} C^{-1} & A \\ 0 & -A^T C A \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} b \\ f - A^T C b \end{bmatrix}.
\]

The pivots of that matrix on the left side start with \(1/c_1, 1/c_2, \ldots, 1/c_m\). Then we get the \(n\) pivots of \(-A^T C A\) which are **negative**, because this matrix is negative definite.

Altogether we are finding a saddle point \((v, w)\) of the energy (quadratic function). The derivative of that quadratic gives our linear equations. The block matrix in those equations has \(m\) positive eigenvalues and \(n\) negative eigenvalues.

16 The least squares equation \(A^T A v = A^T b\) comes from the projection equation \(A^T e = 0\) for the error \(e = b - Av\). Write those two equations in the symmetric saddle point form of Problem 7 (with \(f = 0\)).

In this case \(w = e\) because the weighting matrix is \(C = I\).

**Solution** Ordinary least squares for \(A v = b\) separates the data vector \(b\) in two perpendicular parts:

\[
b = (A\hat{v}) + (b - A\hat{v}) = \text{(projection of } b\text{)} + \text{(error in } b\text{)}.
\]

The error \(e = b - Av\) satisfies \(A^T e = A^T b - A^T A v = 0\) (which means that \(A^T A v = A^T b\), the key equation). That equation \(d^T e = 0\) is Kirchhoff’s Current Law for flows in
a network. It is a candidate for the “most important equation in applied mathematics”—
the conservation equation or continuity equation “flow in = flow out.”

In the form of Problem 15 (with $C = I$) the equations are

$$\begin{bmatrix}
I & A \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
e \\
v
\end{bmatrix} =
\begin{bmatrix}
b \\
0
\end{bmatrix} \quad \text{or} \quad e + Av = b \quad \text{or} \quad A^T e = 0.$$

17 Find the three eigenvalues and three pivots and the determinant of this saddle point
matrix with $C = I$. One eigenvalue is negative because $A$ has one column:

$$m = 2, n = 1 \quad \begin{bmatrix}
C^{-1} & A \\
A^T & 0
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 1 \\
-1 & 1 & 0
\end{bmatrix}.$$

**Solution** The eigenvalues come from $\det(M - \lambda I) = 0$:

$$\begin{bmatrix}
1 - \lambda & 0 & -1 \\
0 & 1 - \lambda & 1 \\
-1 & 1 & -\lambda
\end{bmatrix} = -\lambda(1 - \lambda)^2 - 2(1 - \lambda) = 0.$$

Then $(1 - \lambda)(\lambda^2 - \lambda - 2) = 0$ and $(1 - \lambda)(\lambda - 2)(\lambda + 1) = 0$ and the eigenvalues are

$$\lambda = 1, 2, -1.$$ Check the sum $1 + 2 - 1 = 2$ equal to the trace (sum down the main
diagonal $1 + 1 + 0 = 2$).

The determinant is the product $\lambda_1 \lambda_2 \lambda_3 = (1)(2)(-1) = -2$. Notice $m = 2$ positive
$\lambda$’s and $n = 1$ negative eigenvalue.

Elimination finds the three pivots (which also multiply to give $\det M = -2$):

$$\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 1 \\
-1 & 1 & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 1 \\
0 & 1 & -1
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 1 \\
0 & 0 & -2
\end{bmatrix}. $$
Problem Set 8.1, page 443

1. (a) To prove that \( \cos nx \) is orthogonal to \( \cos kx \) when \( k \neq n \), use \( (\cos nx) (\cos kx) = \frac{1}{2} \cos (n+k)x + \frac{1}{2} \cos (n-k)x \). Integrate from \( x = 0 \) to \( x = \pi \). What is \( \int_0^\pi \cos^2 kx \, dx \)?

(b) **Correction** From 0 to \( \pi \), \( \cos x \) is not orthogonal to \( \sin 2x \) (the book wrongly proposed \( \int_0^\pi \cos x \sin x \, dx \), but this is zero). For orthogonality of all sines and cosines, the period has to be \( 2\pi \).

**Solution** (a) \[
\int_0^\pi (\cos nx)(\cos kx) \, dx = \frac{1}{2} \int_0^\pi \cos (n+k)x \, dx + \frac{1}{2} \int_0^\pi \cos (n-k)x \, dx
\]
\[
= \left[ \frac{\sin (n+k)x}{2(n+k)} + \frac{\sin (n-k)x}{2(n-k)} \right]_0^\pi = 0 + 0
\]

**Solution** (b) \[
\int_0^\pi (\cos x)(\sin 2x) \, dx = \int_0^\pi (\cos x)(2 \sin x \cos x) \, dx = \left[ -\frac{2}{3} \cos^3 x \right]_0^\pi
\]
\[
= \frac{4}{3} \neq 0.
\]

Non-orthogonality comes from \( \int_0^\pi \cos mx \sin nx \, dx \) when \( m - n \) is an odd number.

2. Suppose \( F(x) = x \) for \( 0 \leq x \leq \pi \). Draw graphs for \( -2\pi \leq x \leq 2\pi \) to show three extensions of \( F \): a \( 2\pi \)-periodic even function and a \( 2\pi \)-periodic odd function and a \( \pi \)-periodic function.

**Solution**

3. Find the Fourier series on \( -\pi \leq x \leq \pi \) for
(a) \( f_1(x) = \sin^3 x \), an odd function (sine series, only two terms)

**Solution** (a) The fast way is to know the identity \( \sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x \). This must be the Fourier sine series! It has only two terms.

More slowly, use Euler’s great formula to produce complex exponentials:

\[
(\sin x)^3 = \left( \frac{e^{ix} - e^{-ix}}{2i} \right)^3 = \frac{e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix}}{8i^3} = -\frac{1}{4} \sin 3x + \frac{3}{4} \sin x.
\]

Or slowly compute the usual formulas \( \int \sin^3 x \sin x \, dx \) and \( \int \sin^3 x \sin 3x \, dx \).
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(b) \( f_2(x) = |\sin x| \), an even function (cosine series)

Solution (b)

\[
a_0 = \frac{1}{\pi} \int_{0}^{\pi} |\sin x| \, dx = \frac{2}{\pi}
\]

\[
a_k = \frac{1}{2\pi} \int_{0}^{\pi} |\sin x| \cos kx \, dx = -\frac{1}{4\pi} \left( \frac{\cos(k-1)x}{k-1} + \frac{\cos(k+1)x}{k+1} \right)_{x=0}^{x=\pi}
\]

\[
= 0 \quad \text{(odd } k\text{)} \quad \text{or} \quad -\frac{1}{4\pi} \left[ \frac{-2}{k-1} + \frac{-2}{k+1} \right] = \frac{k}{\pi(k^2-1)} \quad \text{(even } k\text{)}
\]

(c) \( f_3(x) = x \) for \(-\pi \leq x \leq \pi\) (sine series with jump at \( x = \pi \))

Solution (c)

\[
b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin kx \, dx = \left[ \frac{1}{\pi} k^2 \sin kx - \frac{x}{\pi} k \cos kx \right]_{-\pi}^{\pi}
\]

\[
= \frac{1}{\pi} (\cos k\pi + \cos(-k\pi)) = -\frac{2}{k}(-1)^k.
\]

4 Find the complex Fourier series \( e^x = \sum c_k e^{ikx} \) on the interval \(-\pi \leq x \leq \pi\). The even part of a function is \( \frac{1}{2}(f(x) + f(-x)) \), so that \( f_{\text{even}}(x) = f_{\text{even}}(-x) \). Find the cosine series for \( f_{\text{even}} \) and the sine series for \( f_{\text{odd}} \). Notice the jump at \( x = \pi \).

Solution

\[
c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-ikx} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x(1-ik)} \, dx
\]

\[
= \left[ \frac{1}{2\pi(1-ik)} e^{x(1-ik)} \right]_{-\pi}^{\pi} = \frac{e^{\pi(1-ik)} - e^{-\pi(1-ik)}}{2\pi(1-ik)}
\]

The even part of the function is: \( \frac{1}{2}(e^x + e^{-x}) \). The cosine coefficients are

\[
a_0 = \frac{1}{4\pi} \int_{-\pi}^{\pi} (e^x + e^{-x}) \, dx = \frac{1}{2\pi} (e^\pi - e^{-\pi})
\]

\[
a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^x + e^{-x}) \cos kx \, dx = \frac{2k \cosh[\pi] \sin[k\pi] + 2 \cos[k\pi] \sinh[\pi]}{\pi + k^2\pi}
\]

The odd part of the function is: \( \frac{1}{2}(e^x - e^{-x}) \). The sine series is:

\[
b_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^x - e^{-x}) \sin kx \, dx = \frac{2 \cosh[\pi] \sin[k\pi] - 2k \cos[k\pi] \sinh[\pi]}{\pi + k^2\pi}
\]

5 From the energy formula (21), the square wave sine coefficients satisfy

\[
\pi(b_1^2 + b_2^2 + \cdots) = \int_{-\pi}^{\pi} |SW(x)|^2 \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi.
\]
Substitute the numbers $b_k$ from equation (8) to find that $\pi^2 = 8(1 + \frac{1}{9} + \frac{1}{25} + \cdots)$.

**Solution**

The sine coefficients for the odd square wave are

$$b_k = \frac{4}{\pi} \left( \frac{1 - (-1)^k}{2k} \right) = \frac{4}{\pi} \left( \frac{1}{1}, \frac{1}{3}, 0, \frac{1}{5}, 0, \ldots \right)$$

Energy identity gives

$$\pi^2 = 8 \sum_{k=1}^{\infty} \left( \frac{1 - (-1)^k}{2k} \right)^2 = 8 \left( 1 + \frac{1}{9} + \frac{1}{25} + \cdots \right)$$

6. If a square pulse is centered at $x = 0$ to give

$$f(x) = 1\text{ for } |x| < \frac{\pi}{2}, \quad f(x) = 0\text{ for } \frac{\pi}{2} < |x| < \pi,$$

draw its graph and find its Fourier coefficients $a_k$ and $b_k$.

**Solution**

$$a_0 = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} dx = \frac{1}{2}$$

$$a_k = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos kx \, dx = \frac{2}{k\pi} \sin \frac{k\pi}{2} = \sin \left( \frac{k\pi}{2} \right)$$

$$b_k = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \sin kx \, dx = 0$$

7. Plot the first three partial sums and the function $x(\pi - x)$:

$$x(\pi - x) = \frac{8}{\pi} \left( \frac{\sin x}{1} + \frac{\sin 3x}{27} + \frac{\sin 5x}{125} + \cdots \right), 0 < x < \pi.$$ 

Why is $1/k^3$ the decay rate for this function? What is its second derivative?

**Solution**

The parabola $y = x(\pi - x) = x\pi - x^2$ starts at $y(0) = 0$ with slope $y'(0) = \pi$ and second derivative $y''(0) = -2$. Its sine series makes it an odd function $x\pi + x^2$ from $-\pi$ to 0. This odd extension has **second derivative** = $\pm 2$. That jump in $y''$ means that the Fourier coefficients $b_k$ will decay like $1/k^3$. (Remember $1/k$ for jumps in $y(x)$ and $1/k^2$ for jumps in $y'(x)$—no jumps in $y$, $y'$ for this example.)

8. Sketch the $2\pi$-periodic half wave with $f(x) = \sin x$ for $0 < x < \pi$ and $f(x) = 0$ for $-\pi < x < 0$. Find its Fourier series.

**Solution**

The function is not odd or even, so integrals must go from $-\pi$ to $\pi$. The function is zero from $-\pi$ to 0 leaving only these integrals for $a_0, a_k, b_k$:
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\[ a_0 = \frac{1}{2\pi} \int_0^\pi \sin x \, dx = \frac{1}{2\pi} \left[ -\cos x \right]_0^\pi = \frac{1}{\pi} \]

\[ a_k = \frac{1}{\pi} \int_0^\pi \sin x \cos kx \, dx = -\frac{1}{2\pi} \left[ \frac{\cos(1 - k)x}{1 - k} + \frac{\cos(1 + k)x}{1 + k} \right]_0^\pi = \]

\[ \begin{align*}
    \text{[k even]} & \quad \frac{1}{\pi} \left( \frac{1}{1 - k} + \frac{1}{1 + k} \right) = \frac{2}{\pi(1 - k^2)} \quad \text{[and 0 for k odd]} \\
    b_k &= \frac{1}{\pi} \int_0^\pi \sin x \sin kx \, dx \text{ gives } b_1 = \frac{1}{2} \text{ and other } b_k = 0.
\end{align*} \]

9 Suppose \( G(x) \) has period \( 2L \) instead of \( 2\pi \). Then \( G(x + 2L) = G(x) \). Integrals go from \(-L\) to \( L \) or from \( 0 \) to \( 2L \). The Fourier formulas change by a factor \( \pi/L \):

The coefficients in \( G(x) = \sum_{n=-\infty}^{\infty} C_k e^{ik\pi x/L} \) are \( C_k = \frac{1}{2L} \int_{-L}^{L} G(x) e^{-ik\pi x/L} \, dx \).

Derive this formula for \( C_k \): Multiply the first equation for \( G(x) \) by \( e^{-ik\pi x/L} \) and integrate both sides. Why is the integral on the right side equal to \( 2LC_k \)?

**Solution**

Multiply \( G(x) = \sum_{n=-\infty}^{\infty} C_k e^{ik\pi x/L} \) by \( e^{-ik\pi x/L} \). Integrate.

\[ \int_{-L}^{L} G(x) e^{-ik\pi x/L} \, dx = \int_{-L}^{L} e^{-ik\pi x/L} \sum_{n=-\infty}^{\infty} C_k e^{ik\pi x/L} \, dx \]

\[ \int_{-L}^{L} G(x) e^{-ik\pi x/L} \, dx = C_k \int_{-L}^{L} \, dx = 2LC_k \text{ (orthogonality)} \]

\[ C_k = \frac{1}{2L} \int_{-L}^{L} G(x) e^{-ik\pi x/L} \, dx \]

10 For \( G_{\text{even}} \), use Problem 9 to find the cosine coefficient \( A_k \) from \((C_k + C_{-k})/2\):

\[ G_{\text{even}}(x) = \sum_{n=0}^{\infty} A_k \cos \frac{k\pi x}{L} \text{ has } A_k = \frac{1}{L} \int_{0}^{L} G_{\text{even}}(x) \, \cos \frac{k\pi x}{L} \, dx. \]

\( G_{\text{even}} \) is \( \frac{1}{2}(G(x) + G(-x)) \). Exception for \( A_0 = C_0 \): Divide by \( 2L \) instead of \( L \).

**Solution**

The result comes directly from \( \frac{1}{2}(C_k + C_{-k}) \).

11 Problem 10 tells us that \( a_k = \frac{1}{2}(c_k + c_{-k}) \) on the usual interval from \( 0 \) to \( \pi \).

Find a similar formula for \( b_k \) from \( c_k \) and \( c_{-k} \). In the reverse direction, find the complex coefficient \( c_k \) in \( F(x) = \sum c_k e^{ikx} \) from the real coefficients \( a_k \) and \( b_k \).
8.1. Fourier Series

Solution  Solution and correction We are comparing two ways to write a Fourier series:
\[ \sum_{-\infty}^{\infty} c_k e^{ikx} = a_0 + \sum_{1}^{\infty} a_k \cos kx + \sum_{1}^{\infty} b_k \sin kx \]

Pick out the terms for \( k \) and \(-k\):
\[ c_k e^{ikx} + c_{-k} e^{-ikx} = a_k \cos kx + b_k \sin kx \]

Use Euler’s formula to reach cosines/sines on both sides:
\[ (c_k + c_{-k}) \cos kx + i(c_k - c_{-k}) \sin kx = a_k \cos kx + b_k \sin kx \]

This shows that \( a_k = c_k + c_{-k} \) (correction from text) and \( b_k = i(c_k - c_{-k}) \).

Reverse Euler’s formula to reach complex exponentials on both sides:
\[ c_k e^{ikx} + c_{-k} e^{-ikx} = \frac{1}{2} a_k (e^{ikx} + e^{-ikx}) + \frac{1}{2i} b_k (e^{ikx} - e^{-ikx}) \]

This shows that \( c_k = \frac{1}{2} a_k + \frac{1}{2i} b_k \) and \( c_{-k} = \frac{1}{2} a_k - \frac{1}{2i} b_k \).

Real functions with real \( a \)'s and \( b \)'s lead to \( c_{-k} = \overline{c_k} \) (complex conjugates)

12 Find the solution to Laplace’s equation with \( u_0 = \theta \) on the boundary. Why is this the imaginary part of \( 2(z - \frac{z^2}{2} + \frac{z^3}{3} \cdots) = 2 \log(1 + z) \)? Confirm that on the unit circle \( z = e^{i\theta} \), the imaginary part of \( 2 \log(1 + z) \) agrees with \( \theta \).

Solution  The sine series of the odd function \( f(\theta) = \theta \) has coefficients \( b_n = \)
\[ \frac{2}{\pi} \int_{0}^{\pi} \theta \sin n\theta \, d\theta = \frac{2}{\pi} \left[ \frac{1}{n^2} \sin n\theta - \frac{\theta \cos n\theta}{n} \right]_{0}^{\pi} = -\frac{2 \cos n\pi}{n} = 2 \left[ \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{8} \cdots \right] \]

The solution to Laplace’s equation inside the circle has factors \( r^n \):
\[ u(r, \theta) = \sum b_n r^n \sin n\theta = 2r \sin \theta \left[ \frac{\sin \theta}{1} + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots \right] \]
\[ = \text{Im} \left[ 2z - 2z^2 + 2z^3 \cdots \right] = \text{Im}[2 \log(1 + z)]. \]

13 If the boundary condition for Laplace’s equation is \( u_0 = 1 \) for \( 0 < \theta < \pi \) and \( u_0 = 0 \) for \(-\pi < \theta < 0\), find the Fourier series solution \( u(r, \theta) \) inside the unit circle. What is \( u \) at the origin \( r = 0 \)?

Solution  This 0-1 step function \( u_0(\theta) \) equals \( \frac{1}{2} + \frac{1}{2} \) (square wave). Equation (8) of the text gives the Fourier sine series for the square wave:
\[ 0-1 \text{ Step Function} \quad u_0(\theta) = \frac{1}{2} + \frac{2}{\pi} \left[ \frac{\sin \theta}{1} + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots \right] \]

Then the solution to Laplace’s equation includes factors \( r^n \):
\[ u(r, \theta) = \frac{1}{2} + \frac{2}{\pi} \left[ \frac{r \sin \theta}{1} + \frac{r^3 \sin 3\theta}{3} + \frac{r^5 \sin 5\theta}{5} + \cdots \right] = \frac{1}{2} \text{ at } r = 0. \]
14 With boundary values \( u_0(\theta) = 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \cdots \), what is the Fourier series solution to Laplace’s equation in the circle? Sum this geometric series.

**Solution** Inside the circle we see factors \( r^n \) (and \( 1 + x + x^2 + \cdots = 1/(1-x) \)):

\[
u(r, \theta) = 1 + \frac{1}{2}re^{i\theta} + \frac{1}{4}r^2e^{2i\theta} + \cdots = \frac{1}{1 - \frac{1}{2}re^{i\theta}}.
\]

15 (a) Verify that the fraction in Poisson’s formula (30) satisfies Laplace’s equation.

**Solution** (a) We could verify Laplace’s equation in \( r, \theta \) coordinates or recognize that every term in the sum (29) solves that equation:

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.
\]

(b) Find the response \( u(r, \theta) \) to an impulse at \( x = 0, y = 1 \) (where \( \theta = \frac{\pi}{2} \)).

**Solution** (b) When the source is at the point \( \theta = \frac{\pi}{2} \), this replaces \( r \cos \theta \) by \( -r \cos \theta \) in equation (30). Then the response to a point source is infinite at \( r = 1, \theta = \frac{\pi}{2} \):

\[
u(r, \theta) = \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 + 2r \cos \theta}
\]

16 With complex exponentials in \( F(x) = \sum c_k e^{ikx} \), the energy identity (21) changes to

\[
\int_{-\pi}^{\pi} |F(x)|^2 \, dx = 2\pi \sum |c_k|^2.
\]

Derive this by integrating \( (\sum c_k e^{ikx})(\sum \bar{c}_k e^{-ikx}) \).

**Solution** All products \( e^{ikx}e^{-ikx} \) integrate to zero except when \( n = k \):

\[
\int_{-\pi}^{\pi} (c_k e^{ikx})(\bar{c}_k e^{-ikx}) \, dx = 2\pi c_k \bar{c}_k = 2\pi |c_k|^2.
\]

The total energy is the sum over all \( k \).

17 A centered square wave has \( F(x) = 1 \) for \( |x| \leq \pi/2 \).

(a) Find its energy \( \int |F(x)|^2 \, dx \) by direct integration

**Solution** (a) \( \int |F(x)|^2 \, dx = \int_{-\pi/2}^{\pi/2} \, dx = \pi \).

(b) Compute its Fourier coefficients \( c_k \) as specific numbers

**Solution** (b) \( c_k = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-ikx} \, dx = \left[ \frac{1}{2\pi} \frac{e^{-ikx}}{-ik} \right]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi ik} \left( e^{ik\pi/2} - e^{-ik\pi/2} \right) = \frac{1}{\pi k} \sin \left( \frac{k\pi}{2} \right) \)

(c) Find the sum in the energy identity (Problem 8).

**Solution** (c) \( \sin \frac{k\pi}{2} = 1, 0, -1, 0 \) (repeated) so \( 2\pi \sum |c_k|^2 = \frac{2}{\pi} \left( \frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \cdots \right) = 1 \).
18 \( F(x) = 1 + (\cos x)/2 + \cdots + (\cos nx)/2^n + \cdots \) is analytic: infinitely smooth.
(a) If you take 10 derivatives, what is the Fourier series of \( d^{10}F/dx^{10} \)?
(b) Does that series still converge quickly? Compare \( n^{10} \) with \( 2^n \) for \( n = 2^{10} \).

**Solution**

(a) 10 derivatives of \( \cos nx \) gives \( -n^{10}\cos nx \):
\[
\frac{d^{10}F}{dx^{10}} = \frac{1}{2} \cos x - \frac{2^{10}}{2^2} \cos 2x - \frac{3^{10}}{2^3} \cos 3x \cdots - \frac{n^{10}}{2^n} \cos nx - \cdots
\]

(b) Yes, \( 2^n \) gets large much faster than \( n^{10} \) so the series easily converges.
At \( n = 2^{10} = 1024 \) we have \( 2^n = 2^{1024} \), much larger than \( n^{10} = 2^{100} \).

19 If \( f(x) = 1 \) for \(|x| \leq \pi/2\) and \( f(x) = 0 \) for \( \pi/2 < |x| < \pi \), find its cosine coefficients. Can you graph and compute the Gibbs overshoot at the jumps?

**Solution**

\[
a_0 = \text{average value} = \frac{1}{2}
\]

\[
a_k = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos kx \, dx = \left[ \frac{1}{\pi k} \sin kx \right]_{-\pi/2}^{\pi/2} = \frac{2}{\pi k} \sin \frac{k\pi}{2}
\]

20 Find all the coefficients \( a_k \) and \( b_k \) for \( F, I, \) and \( D \) on the interval \(-\pi \leq x \leq \pi\):

\[
F(x) = \delta \left( x - \frac{\pi}{2} \right) \quad I(x) = \int_0^x \delta \left( x - \frac{\pi}{2} \right) \, dx \quad D(x) = \frac{d}{dx} \delta \left( x - \frac{\pi}{2} \right).
\]

**Solution**

(a) Integrate \( \cos kx \) and \( \sin kx \) against \( \delta(x - \frac{\pi}{2}) \) to get
\[
a_0 = \frac{1}{2\pi} \quad a_k = \frac{1}{\pi} \cos \frac{k\pi}{2} \quad \text{and} \quad b_k = \frac{1}{\pi} \sin \frac{k\pi}{2}
\]

(b) The integral \( I(x) \) is the unit step function \( H(x - \frac{\pi}{2}) \) with jump at \( x = \frac{\pi}{2} \):
\[
a_0 = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 1 \, dx = \frac{1}{4}
\]
\[
a_k = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos kx \, dx = \frac{1}{\pi k} \left( \sin k\pi - \sin \frac{k\pi}{2} \right) = -\frac{1}{\pi k} \sin \frac{k\pi}{2}
\]
\[
b_k = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \sin kx \, dx = -\frac{1}{\pi k} \left( \cos k\pi - \cos \frac{k\pi}{2} \right)
\]

(c) \( D(x) \) is the “doublet” = derivative of the delta function \( \delta \left( x - \frac{\pi}{2} \right) \). You must integrate by parts (and \( D(-\pi) = D(\pi) = 0 \) fortunately).
\[
\frac{1}{\pi} \int_{-\pi}^{\pi} D(x) \cos kx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta \left( x - \frac{\pi}{2} \right) (k \sin kx) \, dx
\]

So \( a_k \) for \( D(x) \) is \( kb_k \) **in part (b)**, and \( b_k \) for \( D(x) \) is \( -ka_k \) **in part (b)**.
21 For the one-sided tall box function in Example 4, with \( F = 1/h \) for \( 0 \leq x \leq h \), what is its odd part \( \frac{1}{2}(F(x) - F(-x)) \)? I am surprised that the Fourier coefficients of this odd part disappear as \( h \) approaches zero and \( F(x) \) approaches \( \delta(x) \).

Solution Every function has an even part and an odd part:
\[
F_{\text{even}}(x) = \frac{1}{2}(F(x) + F(-x)) \quad F_{\text{odd}}(x) = \frac{1}{2}(F(x) - F(-x)) \quad F = F_{\text{even}} + F_{\text{odd}}
\]

For the one-sided box function, those even and odd parts are
\[
F_{\text{even}}(x) = \frac{1}{2}h \quad \text{for} \quad |x| \leq h \quad F_{\text{odd}}(x) = \begin{cases} -\frac{1}{h} & \text{for} \quad -h \leq x < 0 \\ + \frac{1}{h} & \text{for} \quad 0 < x \leq h. \end{cases}
\]

The Fourier coefficients of \( F_{\text{odd}} \) don’t really “disappear” as \( h \rightarrow 0 \), because the energy \( \int |F_{\text{odd}}|^2 \, dx \) is growing. But it is growing in the high frequencies and any particular coefficient \( c_k \) (at a fixed frequency \( k \)) approaches zero as \( h \rightarrow 0 \).

22 Find the series \( F(x) = \sum c_k e^{ikx} \) for \( F(x) = e^x \) on \( -\pi \leq x \leq \pi \). That function \( e^x \) looks smooth, but there must be a hidden jump to get coefficients \( c_k \) proportional to \( 1/k \). Where is the jump?

Solution When \( e^x \) is made into a periodic function there is a jump (or a drop) at \( x = \pi \). The drop from \( e^\pi \) to \( e^{-\pi} \) starts the next \( 2\pi \)-interval. That drop shows up as a factor multiplying the \( 1/k \) decay that all jump functions show in their Fourier expansion:
\[
c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-ikx} \, dx = \frac{1}{2\pi} \left[ e^{(1-ik)x} \right]_{x=-\pi}^{\pi} = \frac{1}{2\pi} \frac{e^\pi - e^{-\pi}}{1-ik}.
\]

23 (a) (Old particular solution) Solve \( Ay'' + By' + Cy = e^{ikx} \).

(b) (New particular solution) Solve \( Ay'' + By' + Cy = \sum c_k e^{ikx} \).

Solution This problem shows directly the power of linearity to deal with complicated forcing functions as combinations of simple forcing functions \( e^{ikx} \):
\[
Ay'' + By' + Cy = e^{ikx} \quad \text{has} \quad y_p = \frac{1}{(ik)^2A + ikB + C} e^{ikx} = Y_k e^{ikx} \quad \text{for} \quad k \neq 0.
\]

Problem Set 8.2, page 453

1 Multiply the three matrices in equation (11) and compare with \( F \). In which six entries do you need to know that \( i^2 = -1 \)? This is \( (w_N)^2 = w_2 \). If \( M = N/2 \), why is \( (w_N)^M = -1 \)?

Solution

2 Why is row \( i \) of \( \overline{F} \) the same as row \( N - i \) of \( F \) (numbered from 0 to \( N - 1 \))?

Solution
3 From Problem 8, find the 4 by 4 permutation matrix P so that \( F = P \overline{F} \). Check that \( P^2 = I \) so that \( P = P^{-1} \). Then from \( \overline{F} F = 4I \) show that \( F^2 = 4P \).

It is amazing that \( F^4 = 16 P^2 = 16I \). Four transforms of any \( c \) bring back 16 \( c \). For all \( N \), \( F^2 / N \) is a permutation matrix \( P \) and \( F^4 = N^2 I \).

Solution

4 Invert the three factors in equation (11) to find a fast factorization of \( F^{-1} \).

5 \( F \) is symmetric. Transpose equation (11) to find a new Fast Fourier Transform.

Solution

6 All entries in the factorization of \( F \) involve powers of \( w = \text{sixth root of 1} \):

\[
F_0 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_3 \\ F_3 \end{bmatrix} \begin{bmatrix} P \\ \end{bmatrix} .
\]

Write down these factors with 1, \( w \), \( w^2 \) in \( D \) and powers of \( w^2 \) in \( F_3 \). Multiply!

Solution

7 Put the vector \( c = (1, 0, 1, 0) \) through the three steps of the FFT to find \( y = Fc \). Do the same for \( c = (0, 1, 0, 1) \).

Solution

8 Compute \( y = F_8 c \) by the three FFT steps for \( c = (1, 0, 1, 0, 1, 0, 1, 0) \). Repeat the computation for \( c = (0, 1, 0, 1, 0, 1, 0, 1) \).

Solution

9 If \( w = e^{2\pi i / 64} \) then \( w^2 \) and \( \sqrt{w} \) are among the ____ and ____ roots of 1.

Solution

10 \( F \) is a symmetric matrix. Its eigenvalues aren’t real. How is this possible?

Solution

The three great symmetric tridiagonal matrices of applied mathematics are \( K \), \( B \), \( C \). The eigenvectors of \( K \), \( B \), and \( C \) are discrete sines, cosines, and exponentials. The eigenvector matrices give the DST, DCT, and DFT — discrete transforms for signal processing. Notice that diagonals of the circulant matrix \( C \) loop around to the far corners.

\[
K = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \\ & & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 2 & -1 & & -1 \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \\ & & -1 & 2 \end{bmatrix}, \quad K_{11} = K_{NN} = 2, \quad B_{11} = B_{NN} = 1, \quad C_{1N} = C_{N1} = -1.
\]
11 The eigenvectors of $K_N$ and $B_N$ are the discrete sines $s_1, \ldots, s_N$ and the discrete cosines $c_0, \ldots, c_{N-1}$. Notice the eigenvector $e_0 = (1, 1, \ldots, 1)$. Here are $s_k$ and $e_k$—these vectors are samples of $\sin kx$ and $\cos kx$ from 0 to $\pi$.

$$
\left(\sin\frac{\pi k}{N+1}, \sin\frac{2\pi k}{N+1}, \ldots, \sin\frac{N\pi k}{N+1}\right) \quad \text{and} \quad \left(\cos\frac{\pi k}{2N}, \cos\frac{3\pi k}{2N}, \ldots, \cos\frac{(2N-1)\pi k}{2N}\right)
$$

For 2 by 2 matrices $K_2$ and $B_2$, verify that $s_1$, $s_2$ and $e_0$, $e_1$ are eigenvectors.

Solution

12 Show that $C_3$ has eigenvalues $\lambda = 0, 3, 3$ with eigenvectors $e_0 = (1, 1, 1)$, $e_1 = (1, w, w^2)$, $e_2 = (1, w^2, w)$. You may prefer the real eigenvectors $(1, 1, 1)$ and $(1, 0, -1)$ and $(1, -2, 1)$.

Solution

13 Multiply to see the eigenvectors $e_k$ and eigenvalues $\lambda_k$ of $C_N$. Simplify to $\lambda_k = 2 - 2 \cos(2\pi k/N)$. Explain why $C_N$ is only semidefinite. It is not positive definite.

$$
Ce_k = \left[\begin{array}{ccc}
2 & -1 & 1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{array}\right] \left[\begin{array}{c}
w^k \\
w^{2k} \\
w^{(N-1)k}
\end{array}\right] = (2 - w^k - w^{-k}) \left[\begin{array}{c}
w^k \\
w^{2k} \\
w^{(N-1)k}
\end{array}\right].
$$

Solution

14 The eigenvectors $e_k$ of $C$ are automatically perpendicular because $C$ is a _______ matrix. (To tell the truth, $C$ has repeated eigenvalues as in Problem 12. There was a plane of eigenvectors for $\lambda = 3$ and we chose orthogonal $e_1$ and $e_2$ in that plane.)

Solution

15 Write the 2 eigenvalues for $K_2$ and the 3 eigenvalues for $B_3$. Always $K_N$ and $B_{N+1}$ have the same $N$ eigenvalues, with the extra eigenvalue _______ for $B_{N+1}$. (This is because $K = A^T A$ and $B = AA^T$.)

Solution

---

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1 When the driving function is $f(t) = \delta(t)$, the solution starting from rest is the **impulse response**. The impulse is $\delta(t)$, the response is $y(t)$. Transform this equation to find the **transfer function** $Y(s)$. Invert to find the impulse response $y(t)$.

$$ y'' + y = \delta(t) \text{ with } y(0) = 0 \text{ and } y'(0) = 0 $$

Solution Take the Laplace Transform of $y'' + y = \delta(t)$ with $y(0) = y'(0) = 0$:

$$ s^2Y(s) - sy(0) - y'(0) + Y(s) = 1 $$

$$ Y(s)(s^2 + 1) = 1 $$

$$ Y(s) = \frac{1}{s^2 + 1} \text{ is the transform of } y(t) = \sin t. $$
2 (Important) Find the first derivative and second derivative of \( f(t) = \sin t \) for \( t \geq 0 \).
Watch for a jump at \( t = 0 \) which produces a spike (delta function) in the derivative.

**Solution** The first derivative of \( \sin(t) \) is \( \cos(t) \), and the second derivative is \( -\sin(t) + \delta(t) \).

3 Find the Laplace transform of the unit box function \( b(t) = \{1 \; \text{for} \; 0 \leq t < 1\} = H(t) - H(t - 1) \). The unit step function is \( H(t) \) in honor of Oliver Heaviside.

**Solution** The unit box function is \( f(t) = H(t) - H(t - 1) \)

The transform is \( F(s) = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1}{s}(1 - e^{-s}) \)

The same result comes from \( F(s) = \int_{0}^{\infty} f(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt \).

4 If the Fourier transform of \( f(t) \) is defined by \( \hat{f}(k) = \int f(t)e^{-ikt}dt \) and \( f(t) = 0 \) for \( t < 0 \), what is the connection between \( \hat{f}(k) \) and the Laplace transform \( F(s) \)?

**Solution** The Fourier Transform is the Laplace Transform with \( s = ik \): \( \hat{f}(k) = F(ik) \).

5 What is the Laplace transform \( R(s) \) of the standard ramp function \( r(t) = t \)? For \( t < 0 \) all functions are zero. The derivative of \( r(t) \) is the unit step \( H(t) \). Then multiplying \( R(s) \) by \( s \) gives \( \frac{1}{s} \).

**Solution** The Laplace Transform \( R(s) \) of the Ramp Function \( r(t) = t \) is
\[
R(s) = \int_{0}^{\infty} te^{-st}dt = -\frac{te^{-st}}{s}\bigg|_{0}^{\infty} - \int_{0}^{\infty} \frac{e^{-st}}{s}dt = 0 - \frac{e^{-st}}{s^{2}}\bigg|_{0}^{\infty} = \frac{1}{s^{2}}
\]

Multiplying \( R(s) \) by \( s \) gives the Laplace transform \( \frac{1}{s} \) of the step function.

6 Find the Laplace transform \( F(s) \) of each \( f(t) \), and the poles of \( F(s) \):
   (a) \( f = 1 + t \) \quad (b) \( f = t \cos \omega t \) \quad (c) \( f = \cos(\omega t - \theta) \)
   (d) \( f = \cos^{2} t \) \quad (e) \( f = e^{-2t} \cos t \) \quad (f) \( f = te^{-t} \sin \omega t \)

**Solution (a)** The transform of \( f(t) = 1 + t \) has a **double pole** at \( s = 0 \):
\[
F(s) = \int_{0}^{\infty} (1 + t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt + \int_{0}^{\infty} te^{-st}dt = \frac{1}{s} + \frac{1}{s^{2}} = \frac{1 + s}{s^{2}}
\]

**Solution (b)**

\[
f(t) = t \cos(\omega t) = t \left( \frac{e^{i\omega t} + e^{-i\omega t}}{2} \right) = \frac{te^{i\omega t}}{2} + \frac{te^{-i\omega t}}{2}
\]
transforms to
\[
F(s) = \int_{0}^{\infty} \frac{te^{(i\omega - s)t}}{2}dt + \int_{0}^{\infty} \frac{te^{-((i\omega - s)t}}{2}dt
\]
\[
= \frac{-e^{-(s-i\omega)(st-it\omega + 1)}}{2(s-i\omega)^{2}}\bigg|_{0}^{\infty} + \frac{-e^{-(s+i\omega)(st+it\omega + 1)}}{2(s+i\omega)^{2}}\bigg|_{0}^{\infty}
\]
\[
= \frac{1}{2(s-i\omega)^{2}} + \frac{1}{2(s+i\omega)^{2}} = \frac{(s-i\omega)^{2} + (s+i\omega)^{2}}{2(s-i\omega)^{2}(s+i\omega)^{2}} = \frac{s^{2} - \omega^{2}}{(s^{2} + \omega^{2})^{2}}
\]
Poles occur at \( s = i\omega \) and \( s = -i\omega \), the two exponents of \( f(t) \).
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Solution (c) \( f(t) = \cos(\omega t - \theta) = \cos \omega t \cos \theta + \sin \omega t \sin \theta \) transforms to

\[
F(s) = \frac{s}{s^2 + \omega^2} \cos \theta + \frac{\omega}{s^2 + \omega^2} \sin \theta
\]

Poles occur at \( s = \pm i \omega \).

Solution (d)

\[
f(t) = \cos^2(t) = \frac{1}{4}(e^{i t} + e^{-it})^2 = \frac{1}{4}(e^{2it} + 2 + e^{-2it})
\]

\[
F(s) = \int_0^\infty \frac{1}{4}(e^{2it} + e^{-2it} + 2)e^{-st} dt
\]

\[
= -\frac{1}{4(s-2t)} + \frac{1}{4(s+2t)} + \frac{1}{2s} = \frac{2s}{4(s^2 + 4)} + \frac{1}{2s} = \frac{s^2 + 2}{s(s^2 + 4)}
\]

Poles occur at \( s = 0 \) and \( s = \pm 2i \). Another way is to write \( \cos^2 t = \frac{1 + \cos 2t}{2} \)

Solution (e)

\[
f(t) = e^{-2t} \cos t = \frac{1}{2}e^{(i-2)t} + \frac{1}{2}e^{-(i+2)t}
\]

\[
F(s) = \int_0^\infty \frac{1}{2}e^{(i-2)t}e^{-st} dt + \int_0^\infty \frac{1}{2}e^{-(i+2)t}e^{-st} dt
\]

\[
= \frac{1}{2(-i + 2 + s)} + \frac{1}{2(i + 2 + s)} = \frac{s + 2}{(s + 2)^2 + 1}
\]

Poles occur at the exponents \( s = -2 \pm i \) in \( f(t) \).

Solution (f)

\[
f(t) = te^{-t} \sin \omega t = \frac{t}{2i}e^{(i\omega-1)t} - \frac{t}{2i}e^{-(i\omega+1)t}
\]

\[
F(s) = \int_0^\infty \left( \frac{t}{2i}e^{(i\omega-1)t} - \frac{t}{2i}e^{-(i\omega+1)t} \right) e^{-st} dt
\]

\[
= \left[ \frac{t}{2i}(e^{(i\omega-1)s} - e^{-(i\omega+1)s}) \right]_0^\infty - \frac{i(1 + t(s - i\omega + 1))}{2(s - i\omega + 1)^2} - \frac{i(1 + t(s + i\omega + 1))}{2(s + i\omega + 1)^2}
\]

Poles of \( F(s) \) occur at \( s = -1 \pm i \omega \), the exponents of \( f(t) \).

7 Find the Laplace transform \( s \) of \( f(t) = \) next integer above \( t \) and \( f(t) = t \delta(t) \).

A staircase \( f(t) = [t] = H(t) + H(t - 1) + H(t - 2) + \cdots = \) next integer above \( t \) is a sum of step functions. The transform is

\[
\frac{1}{s} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \cdots = \frac{1}{s} \left( 1 + e^{-s} + e^{-2s} + \cdots \right) = \frac{1}{s} \left( \frac{1}{1 - e^{-s}} \right).
\]

The differentiation rule \( \mathcal{L}(tf(t)) = -F'(s) \) with \( f(t) = \delta(t) \) and \( F(s) = 1 \) gives

\[
\mathcal{L}(t\delta(t)) = -\frac{d}{ds}(1) = 0 \quad \text{(this is correct because } t\delta(t) \text{ is the zero function).}
\]
8.5. The Laplace Transform

8 Inverse Laplace Transform: Find the function \( f(t) \) from its transform \( F(s) \):

(a) \( \frac{1}{s - 2\pi i} \)

(b) \( \frac{s + 1}{s^2 + 1} \)

(c) \( \frac{1}{(s - 1)(s - 2)} \)

(d) \( \frac{1}{s^2 + 2s + 10} \)

(e) \( e^{-s}/(s - a) \)

(f) \( 2s \)

Solution (a) \( F(s) = \frac{1}{s - 2\pi i} \) is the transform of \( f(t) = e^{2\pi it} \).

Solution (b) \( F(s) = \frac{s}{s^2 + 1} + \frac{1}{s^2 + 1} \) is the transform of \( f(t) = \cos t + \sin t \).

Solution (c) \( F(s) = \frac{1}{(s - 1)(s - 2)} = \frac{1}{s - 2} - \frac{1}{s - 1} \) is the transform of \( f(t) = e^{2t} - e^t \).

Solution (d)

\[
F(s) = \frac{1}{s^2 + 2s + 10} = \frac{1}{(s + 1 + 3i)(s + 1 - 3i)}
\]

\[
= \frac{i}{6(s + (1 + 3i))} - \frac{i}{6(s + (1 - 3i))}
\]

\[
f(t) = \frac{i}{6} e^{-(1+3i)t} - \frac{i}{6} e^{-(1-3i)t}
\]

\[
= -\frac{e^{-t} \sin(3t)}{3}
\]

Solution (e)

\[F(s) = \frac{e^{-s}}{s - a}\]

\[f(t) = e^{at}H(t - 1) = \text{shift of } e^{at}\]

Solution (f)

\[F(s) = 2s\]

\[f(t) = 2 \delta/dt\]

9 Solve \( y'' + y = 0 \) from \( y(0) \) and \( y'(0) \) by expressing \( Y(s) \) as a combination of \( s/(s^2 + 1) \) and \( 1/(s^2 + 1) \). Find the inverse transform \( y(t) \) from the table.

Solution

\[y'' + y = 0\]

\[s^2Y(s) - sy(0) - y'(0) + Y(s) = 0\]

\[Y(s)(s^2 + 1) = sy(0) + y'(0)\]

\[Y(s) = y(0)\frac{s}{s^2 + 1} + y'(0)\frac{1}{s^2 + 1}\]

The inverse transform is \( y(t) = y(0)\cos(t) + y'(0)\sin(t) \).

10 Solve \( y'' + 3y' + 2y = \delta \) starting from \( y(0) = 0 \) and \( y'(0) = 1 \) by Laplace transform. Find the poles and partial fractions for \( Y(s) \) and invert to find \( y(t) \).

Solution The transform of \( \frac{d^2y}{dt^2} + 3 \frac{dy}{dt} + 2y = \delta(t) \) with \( y(0) = 0 \) and \( y'(0) = 1 \) is
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\[ s^2Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0) + 2Y(s) = 1 \]

\[ Y'(s)(s^2 + 3s + 2) - 1 = 1 \]

\[ Y(s) = \frac{2}{(s + 1)(s + 2)} \]

\[ Y(s) = \frac{2}{s + 1} - \frac{2}{s + 2} \]

\[ y(t) = 2e^{-t} - 2e^{-2t} \]

11 Solve these initial-value problems by Laplace transform:

(a) \[ y' + y = e^{i\omega t}, y(0) = 8 \]

(b) \[ y'' - y = e^t, y(0) = 0, y'(0) = 0 \]

(c) \[ y' + y = e^{-t}, y(0) = 2 \]

(d) \[ y'' + y = 6t, y(0) = 0, y'(0) = 0 \]

(e) \[ y' - i\omega y = \delta(t), y(0) = 0 \]

(f) \[ my'' + cy' + ky = 0, y(0) = 1, y'(0) = 0 \]

Solution (a)

\[ y' + y = e^{i\omega t} \text{ with } y(0) = 8 \]

\[ sY(s) - 8 + Y(s) = \frac{1}{s - i\omega} + 8 \]

\[ Y(s)(s + 1) = \frac{1}{s - i\omega} + 8 \]

\[ Y(s) = \frac{1}{(s + 1)(s - i\omega)} + \frac{8}{s + 1} \]

\[ Y(s) = \frac{1}{1 + i\omega} \left( \frac{1}{s - i\omega} - \frac{1}{s + 1} \right) + \frac{8}{s + 1} \]

Particular + null \[ y(t) = \frac{1}{1 + i\omega} \left( e^{i\omega t} - e^{-t} \right) + 8e^{-t} \]

Solution (b)

\[ y'' - y = e^t \text{ with } y(0) = 0 \text{ and } y'(0) = 0 \]

\[ s^2Y(s) - Y(s) = \frac{1}{s - 1} \]

\[ Y(s) = \frac{1}{4(s + 1)^2} - \frac{1}{4(s - 1)} + \frac{1}{2(s - 1)^2} \]

\[ y(t) = \frac{e^{-t}}{4} - \frac{e^t}{4} + \frac{te^t}{2} \]

Solution (c)

\[ y' + y = e^{-t} \text{ with } y(0) = 2 \]

\[ sY(s) - 2 + Y(s) = \frac{1}{s + 1} \]

\[ Y(s) = \frac{1}{(s + 1)^2} + \frac{2}{s + 1} \]

\[ y(t) = te^{-t} + 2e^{-t} \]

Solution (d)
8.5. The Laplace Transform

\[ y'' + y = 6t \quad \text{with} \quad y(0) = y'(0) = 0 \]

\[ s^2 Y(s) + Y(s) = \frac{6}{s^2} \]

\[ Y(s)(s^2 + 1) = \frac{6}{s^2} \]

\[ Y(s) = \frac{s^2}{s^2 + 1} - \frac{3i}{s + i} + \frac{3i}{s - i} \]

\[ y(t) = 6t - 3ie^{-it} + 3ie^{it} = 6t - 6\sin t \]

**Solution (e)**

\[ y' - i\omega y = \delta(t) \quad \text{with} \quad y(0) = 0 \]

\[ sY(s) - i\omega Y(s) = 1 \]

\[ Y(s) = \frac{1}{s - i\omega} \]

\[ y(t) = e^{\omega t} \]

**Solution (f)**

\[ my'' + cy' + ky = 0 \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0 \]

\[ ms^2 Y(s) - msy(0) + csY(s) - cy(0) + kY(s) = 0 \]

\[ Y(s)(ms^2 + cs + k) = ms + c \]

\[ Y(s) = \frac{ms + c}{ms^2 + cs + k} \]

has the form \[ \frac{a}{s - s_1} + \frac{b}{s - s_2} \]

We used this Mathematica command to find \( y(t) \)

**Simplify** InverseLaplaceTransform \[ (ms + c)/(ms^2 + cs + k), s, t) \]

\[ y(t) = e^{\frac{\sqrt{c^2 - 4km} + \frac{\sqrt{c^2 - 4km}}{m} + 1 + e^{-\frac{\sqrt{c^2 - 4km}}{m}}}{2\sqrt{c^2 - 4km}}} \]

\[ \text{12 The transform of } e^{At} \text{ is } (sI - A)^{-1}. \text{ Compute that matrix (the transfer function) when } A = [1 \ 1; 1 \ 1]. \text{ Compare the poles of the transform to the eigenvalues of } A. \]

**Solution** When \( A = [1 \ 1; 1 \ 1] \) we have:

\[ (sI - A)^{-1} = \left[ \begin{array}{cc} s - 1 & -1 \\ -1 & s - 1 \end{array} \right]^{-1} = \frac{1}{s^2 - 2s} \left[ \begin{array}{cc} s - 1 & 1 \\ 1 & s - 1 \end{array} \right]. \]

The poles of the system are \( s = 2 \) and \( s = 0 \), the eigenvalues of \( A \).

\[ \text{13 If } dy/dt \text{ decays exponentially, show that } sY(s) \to y(0) \text{ as } s \to \infty. \]

**Solution**

\[ sY(s) = \int_0^\infty se^{-st} y(t) dt \quad \text{(integrate by parts)} \]

\[ = \int_0^\infty e^{-st} \frac{dy}{dt} dt - \left[ e^{-st} y(t) \right]_0^\infty \]

\[ = \int_0^\infty e^{-st} \frac{dy}{dt} dt + y(0) \to y(0) \quad \text{as} \quad s \to \infty \]

**Example:** \[ \frac{dy}{dt} = e^{0.1t} \] has \( sY(s) - y(0) = \frac{1}{s + a} \to 0 \) as \( s \to \infty \).
14 Transform Bessel’s time-varying equation \( ty'' + y' + ty = 0 \) using \( \mathcal{L}[ty] = -dY/ds \) to find a first-order equation for \( Y \). By separating variables or by substituting \( Y(s) = C/\sqrt{1 + s^2} \), find the Laplace transform of the Bessel function \( y = J_0 \).

**Solution** The transform of \( ty'' \) applies the \( \mathcal{L}(t, y') \) rule to \( y'' \) instead of \( y' \):

\[
\mathcal{L}(t, y'') = -\frac{d}{ds}(\text{transform of } y'') = -\frac{d}{ds}(s^2 Y(s) - sy(0) - y'(0)).
\]

Apply this to the transform of \( \frac{d^2 y}{dt^2} + \frac{dy}{dt} + ty = 0 \):

\[
-2sY(s) - s^2 \frac{dY}{ds} + y(0) + sY(s) - y(0) - \frac{dY}{ds} = 0
\]

\[
-sY(s) - s^2 \frac{dY}{ds} - \frac{dY}{ds} = 0
\]

\[
sY(s) = -(s^2 + 1) \frac{dY}{ds}
\]

\[
\frac{dY}{Y(s)} = -\frac{s ds}{s^2 + 1}
\]

\[
\log Y(s) = \log \left( \frac{1}{\sqrt{s^2 + 1}} \right)
\]

The transform of the Bessel solution \( y = J_0 \) is \( Y(s) = \frac{1}{\sqrt{s^2 + 1}} \).

15 Find the Laplace transform of a single arch of \( f(t) = \sin \pi t \).

**Solution** A single arch of \( \sin \pi t \) extends from \( t = 0 \) to \( t = 1 \):

\[
F(s) = \int_0^\infty f(t) e^{-st} dt = \int_0^1 \sin(\pi t) e^{-st} dt = \int_0^1 \frac{e^{i\pi t - st}}{2i} dt - \int_0^1 \frac{e^{-i\pi t - st}}{2i} dt
\]

\[
= \left[ \frac{e^{i\pi t - st}}{2i(\pi - s)} + \frac{e^{-i\pi t - st}}{2i(\pi + s)} \right]_{t=0}^{t=1}
\]

\[
= \frac{e^{i\pi - s} - 1}{2i(\pi - s)} + \frac{e^{-i\pi - s} - 1}{2i(\pi + s)}
\]

\[
= \left( \frac{-e^{-s} - 1}{2i} \right) \left( \frac{1}{i\pi - s} - \frac{1}{i\pi + s} \right) = \left( \frac{e^{-s} + 1}{i} \right) \left( \frac{s}{\pi^2 + s^2} \right)
\]

A faster and more direct approach: One arch of the sine curve agrees with \( \sin \pi t + \text{unit shift of } \sin \pi t \), because those cancel after one arch.

\[
\sin \pi t + \sin \pi (t - 1) = \sin \pi t + \sin \pi t \cos \pi = \sin \pi t - \sin \pi t = 0.
\]

16 Your acceleration \( v' = c(v^* - v) \) depends on the velocity \( v^* \) of the car ahead:

(a) Find the ratio of Laplace transforms \( V^*(s)/V(s) \).

(b) If that car has \( v^* = t \) find your velocity \( v(t) \) starting from \( v(0) = 0 \).

**Solution** (a) Take the Laplace Transform of \( \frac{dv}{dt} = c(v^* - v) \) assuming \( v(0) = 0 \);
8.5. The Laplace Transform

\[ sV(s) - v(0) = cV^*(s) - cV(s) \]
\[ V(s)(s + c) = cV^*(s) \]
\[ \frac{V^*(s)}{V(s)} = \frac{s + c}{c} \]

Solution (b) If \( v^*(t) = t \) then \( V^*(s) = \frac{1}{s^2} \). Therefore

\[ V(s)(s + c) = \frac{c}{s^2} \]
\[ V(s) = \frac{c}{s^3 + cs^2} \]
\[ = \frac{1}{c(s + c)} - \frac{1}{cs} + \frac{1}{s^2} \]
\[ v(t) = e^{-ct} - \frac{1}{c} + t \]

17 A line of cars has \( v_n' = c[v_{n-1}(t - T) - v_n(t - T)] \) with \( v_0(t) = \cos \omega t \) in front.

(a) Find the growth factor \( A = 1/(1 + i\omega e^{i\omega T}/c) \) in oscillation \( v_n = A^n e^{i\omega t} \).

(b) Show that \( |A| < 1 \) and the amplitudes are safely decreasing if \( cT < \frac{1}{2} \).

(c) If \( cT > \frac{1}{2} \) show that \( |A| > 1 \) (dangerous) for small \( \omega \). (Use \( \sin \theta < \theta \).)

Human reaction time is \( T \geq 1 \) sec and human aggressiveness is \( c = 0.4/\text{sec} \).

Danger is pretty close. Probably drivers adjust to be barely safe.

Solution (a) \( \frac{dv_n}{dt} = c[v_{n-1}(t - T) - v_n(t - T)] \) with \( v_n = A^n e^{i\omega t} \)

\[ i\omega A^n e^{i\omega t} = cA^{n-1} e^{i\omega (t-T)} - cA^n e^{i\omega (t-T)} \]
\[ A \frac{i\omega e^{i\omega T}}{c} = 1 - A \]
\[ A \left( 1 + \frac{i\omega e^{i\omega T}}{c} \right) = 1 \]

Solution (b)
Chapter 8. Fourier and Laplace Transforms

For $|A| < 1$ we need $\left| 1 + \frac{i\omega}{c}e^{i\omega T} \right| > 1$

$$\left| 1 - \frac{\omega}{c} \sin(\omega T) + \frac{\omega}{c} \cos(\omega T) \right| > 1$$

$$\left( 1 - \frac{\omega}{c} \sin(\omega T) \right)^2 + \frac{\omega^2}{c^2} \cos^2(\omega T) > 1$$

$$1 - \frac{2\omega}{c} \sin(\omega T) + \frac{\omega^2}{c^2} \sin^2(\omega T) + \frac{\omega^2}{c^2} \cos^2(\omega T) > 1$$

$$1 - \frac{2\omega}{c} \sin(\omega T) + \frac{\omega^2}{c^2} > 1$$

Since $\sin \omega T < \omega T$, we are safe if $\frac{\omega^2}{c^2} > \frac{2\omega}{c} \omega T$ which is $cT < \frac{1}{2}$

**Solution**

Since $\sin \omega T \approx \omega T$ when this number is small. Then the same steps show $|A| > 1$ when $cT > \frac{1}{2}$.

18 For $f(t) = \delta(t)$, the transform $F(s) = 1$ is the limit of transforms of tall thin box functions $b(t)$. The boxes have width $\epsilon \to 0$ and height $1/\epsilon$ and area 1.

Inside integrals, $b(t) = \begin{cases} 1/\epsilon & \text{for } 0 \leq t < \epsilon \\ 0 & \text{otherwise} \end{cases}$ approaches $\delta(t)$.

Find the transform $B(s)$, depending on $\epsilon$. Compute the limit of $B(s)$ as $\epsilon \to 0$.

**Solution**

We begin by finding the transform of the box:

$$B(s) = \int_0^\epsilon \frac{1}{\epsilon} e^{-st} dt = \left[ -\frac{1}{s\epsilon} e^{-st} \right]_0^\epsilon = \frac{1 - e^{-s\epsilon}}{s\epsilon}$$

We take the limit as $\epsilon \to 0$—the box approaches a delta function!

$$B_\epsilon(s) = \lim_{\epsilon \to 0} \frac{1 - e^{-s\epsilon}}{s\epsilon} = \lim_{\epsilon \to 0} \frac{1 - (1 - s\epsilon + \frac{1}{2}s^2\epsilon^2 - \cdots)}{s\epsilon} = 1.$$ 

19 The transform $1/s$ of the unit step function $H(t)$ comes from the limit of the transforms of short steep ramp functions $r_\epsilon(t)$. These ramps have slope $1/\epsilon$:

Compute $R_\epsilon(s) = \int_0^\epsilon \frac{t}{\epsilon} e^{-st} dt + \int_{\epsilon}^{\infty} e^{-st} dt$. Let $\epsilon \to 0$.

**Solution**

$$R_\epsilon(s) = \int_0^\epsilon \frac{t}{\epsilon} e^{-st} dt + \int_{\epsilon}^{\infty} e^{-st} dt = \left[ \frac{e^{-st}(-st - 1)}{es^2} \right]_{t=\epsilon}^{t=\infty} + \left[ \frac{e^{-st}}{-s} \right]_{t=\epsilon}^{t=\infty}$$

$$= \frac{e^{-s\epsilon}(-s\epsilon - 1)}{es^2} + \frac{e^{-s\epsilon}}{s} = \frac{1 - e^{-s\epsilon}}{es^2}$$

$$\lim R_\epsilon(s) = \lim \frac{1 - (1 - s\epsilon + \frac{1}{2}s^2\epsilon^2 - \cdots)}{es^2} = \frac{1}{s}.$$
20 In Problems 18 and 19, show that the derivative of the ramp function \( r_\varepsilon(t) \) is the box function \( b(t) \). The “generalized derivative” of a step is the _______ function.

**Solution** The generalized derivative of the short ramp \( r_\varepsilon(t) \) is the thin box \( b(\varepsilon t) \). We say “generalized” because this is not a true derivative at \( t = 0 \): the ramp has zero slope left of \( t = 0 \) and nonzero slope right of \( t = 0 \). But the transforms of \( r_\varepsilon \) and \( b_\varepsilon \) follow the rule for derivatives.

The generalized derivative of a step function is a delta function.

21 What is the Laplace transform of \( y'''(t) \) when you are given \( Y(s) \) and \( y(0), y'(0), y''(0) \)?

**Solution** The Laplace Transform of \( y'''(t) \) is \( s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \).

22 The Pontryagin maximum principle says that the optimal control is “bang-bang”—it only takes on the extreme values permitted by the constraints. To go from rest at \( x = 0 \) to rest at \( x = 1 \) in minimum time, use maximum acceleration \( A \) and deceleration \( -B \). At what time \( t \) do you change from the accelerator to the brake? (This is the fastest driving between two red lights.)

**Solution** The maximum principle requires full acceleration \( A \) to an unknown time \( t_0 \) and then full deceleration \( -B \) to reach \( x = 1 \) with zero velocity. The velocities are

\[
\begin{align*}
v &= At \quad \text{for } t \leq t_0 \\
v &= At_0 - B(t - t_0) \quad \text{for } t > t_0
\end{align*}
\]

Integrating the velocity \( v = dx/dt \) gives the distance \( x(t) \):

\[
\begin{align*}
x &= \frac{1}{2}At^2 \quad \text{for } t < t_0 \\
x &= \frac{1}{2}At_0^2 \quad \text{at } t = t_0 \\
x &= \frac{1}{2}At_0^2 + At_0(t - t_0) - \frac{1}{2}B(t - t_0)^2 \quad \text{for } t > t_0
\end{align*}
\]

At the final time \( T \) we reach \( x = 1 \) with velocity \( v = 0 \). This gives two equations for \( t_0 \) and \( T \):

\[
\begin{align*}
v &= At_0 - B(T - t_0) = 0 \\
x &= At_0T - \frac{1}{2}At_0^2 - \frac{1}{2}B(T - t_0)^2 = 1
\end{align*}
\]

Substitute \( T = \frac{1}{B}t_0(A + B) \) from the first equation into the second equation. This leaves an ordinary quadratic equation to solve for \( t_0 \).

**Problem Set 8.6, page 453**

1 Find the convolution \( v \ast w \) and also the cyclic convolution \( v \circ w \):

(a) \( v = (1, 2) \) and \( w = (2, 1) \)

**Solution (a)**

Convolution: \( (1, 2) \ast (2, 1) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} \)

Cyclic Convolution: \( (1, 2) \circ (2, 1) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \)
(b) \( v = (1, 2, 3) \) and \( w = (4, 5, 6) \).

**Solution (b)**

\[
(1, 2, 3) * (4, 5, 6) = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \\ 28 \\ 27 \\ 18 \end{bmatrix}
\]

**Cyclic Convolution**:

\[
\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 31 \\ 31 \\ 28 \end{bmatrix}
\]

2 Compute the convolution \((1, 3, 1) * (2, 2, 3) = (a, b, c, d, e)\). To check your answer, add \(a + b + c + d + e\). That total should be 35 since \(1 + 3 + 1 = 5\) and \(2 + 2 + 3 = 7\) and \(5 \times 7 = 35\).

**Solution**

\[
\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 11 \\ 11 \\ 3 \end{bmatrix}
\]

1 + 3 + 1 times 2 + 2 + 3 is 2 + 8 + 11 + 11 + 3 : (5)(7) = (35).

3 Multiply 1 + 3x + x^2 times 2 + 2x + 3x^2 to find \(a + bx + cx^2 + dx^3 + ex^4\). Your multiplication was the same as the convolution \((1, 3, 1) * (2, 2, 3)\) in Problem 8. When \(x = 1\), your multiplication shows why 1 + 3 + 1 = 5 times 2 + 2 + 3 = 7 agrees with \(a + b + c + d + e = 35\).

**Solution**

\[
(1 + 3x + x^2) \times (2 + 2x + 3x^2) = 2 + 2x + 3x^2 + 6x + 6x^2 + 9x^3 + 2x^2 + 2x^3 + 3x^4
\]

\[
= 2 + 8x + 11x^2 + 11x^3 + 3x^4
\]

At \(x = 1\) this is again \((5) \times (7) = (35)\).

4 (Deconvolution) Which vector \(v\) would you convolve with \(w = (1, 2, 3, 0)\) to get \(v * w = (0, 1, 2, 3, 0)\)? Which \(v\) gives \(v \odot w = (3, 1, 2)\)?

**Solution**

\[
\begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}
\]

The first and last equation give \(v_0 = v_2 = 0\). Substituting into the second, third, fourth equation gives \(v_1 = 1\). Therefore \(v = (0, 1, 0)\).

For cyclic convolution

\[
\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_0 & v_2 & v_1 \\ v_1 & v_0 & v_2 \\ v_2 & v_1 & v_0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}
\]

\[
\begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\]
8.6. Convolution (Fourier and Laplace)

5 (a) For the periodic functions \( f(x) = 4 \) and \( g(x) = 2 \cos x \), show that \( f \ast g \) is zero (the zero function)!

**Solution** (a) From equation (4) we have

\[
(f \ast g)(x) = \int_0^{2\pi} g(y)f(x-y) \, dy = 4 \int_0^{2\pi} 2 \cos y \, dy = 4 \cdot 0 = 0 \text{ for all } x.
\]

(b) In frequency space \((k\text{-space})\) you are multiplying the Fourier coefficients of 4 and \( 2 \cos x \). Those coefficients are \( c_0 = 4 \) and \( d_1 = d_{-1} = 1 \). Therefore every product \( c_kd_k \) is _____.

**Solution** (b) In frequency space \((k\text{-space})\) you are multiplying the Fourier coefficients of 4 and \( 2 \cos x \). Those coefficients are \( c_0 = 4 \) and \( d_1 = d_{-1} = 1 \). **Therefore every product \( c_kd_k \) is zero.** These are the coefficients of the zero function.

6 For periodic functions \( f = \sum c_k e^{ikx} \) and \( g = \sum d_k e^{ikx} \), the Fourier coefficients of \( f \ast g \) are \( 2\pi c_k d_k \). Test this factor \( 2\pi \) when \( f(x) = 1 \) and \( g(x) = 1 \) by computing \( f \ast g \) from its definition (6.4).

**Solution** From equation (4):

\[
(f \ast g)(x) = \int_0^{2\pi} f(y)g(x-y) \, dy = \int_0^{2\pi} 1 \cdot 1 \, dy = 2\pi.
\]

The same convolution in \(k\text{-space}\) has \( c_0 = 1 \) and \( d_0 = 1 \) (all other \( c_k = d_k = 0 \)). Then \( 2\pi c_kd_k \) gives the correct coefficients \((2\pi \text{ and } 0)\) of the convolution \( f \ast g \) (which equals \( 2\pi \)).

7 Show by integration that the periodic convolution \( \int_0^{2\pi} \cos x \cos(t-x) \, dx \) is \( \pi \cos t \). In \(k\text{-space}\) you are squaring Fourier coefficients \( c_1 = c_{-1} = \frac{1}{2} \) to get \( \frac{1}{2} \) and \( \frac{1}{2} \); these are the coefficients of \( \frac{1}{2} \cos t \). The \( 2\pi \) in Problem 8 makes \( \pi \cos t \) correct.

**Solution**

\[
\int_0^{2\pi} \cos x \cos(t-x) \, dx = \int_0^{2\pi} \cos x (\cos t \cos x + \sin t \sin x) \, dx = \pi \cos t + 0.
\]

8 Explain why \( f \ast g \) is the same as \( g \ast f \) (periodic or infinite convolution).

**Solution** In Fourier space convolution \( f \ast g \) or \( f \oplus g \) leads to multiplication \( c_kd_k \), which is certainly the same as \( d_kc_k \). So \( f \oplus g = g \oplus f \) in \(x\text{-space}\).

9 What 3 by 3 circulant matrix \( C \) produces cyclic convolution with the vector \( c = (1, 2, 3) \)? Then \( Cd \) equals \( c \oplus d \) for every vector \( d \). Compute \( c \oplus d \) for \( d = (0, 1, 0) \).

**Solution** The circulant matrix \( C = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \) gives cyclic convolution with \( (1, 2, 3) \).

When \( d = (0, 1, 0) \) we have \( c \oplus d = Cd = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \).
10 What 2 by 2 circulant matrix \( C \) produces cyclic convolution with \( e = (1, 1) \)? Show in four ways that this \( C \) is not invertible. Deconvolution is impossible.

(1) Find the determinant of \( C \).
(2) Find the eigenvalues of \( C \).
(3) Find \( d \) so that \( Cd = c \conv d \) is zero.
(4) \( Fc \) has a zero component.

**Solution** The 2 by 2 circulant matrix \( C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \) gives \((1, 1) \conv d = Cd\).

11 (a) Change \( b(x) \ast \delta(x - 1) \) to a multiplication \( \hat{b}(k) \hat{d}(k) \):

The box \( b(x) = \{1 \text{ for } 0 \leq x \leq 1\} \) transforms to \( \hat{b}(k) = \frac{1}{2} e^{-ikx} dx \).

The shifted delta transforms to \( \hat{d}(k) = \int \delta(x - 1)e^{-ikx} dx \).

(b) Show that your result \( \hat{b} \hat{d} \) is the transform of a shifted box function. This shows how convolution with \( \delta(x - 1) \) shifts the box.

**Solution** This question shows that continuous convolution with \( \delta(x - 1) \) produces a shift in the box function \( b(x) \), just like discrete convolution with the shifted delta vector \((\ldots, 0, 0, 1, \ldots)\) produces a one-step shift.

We compute \( \delta(x - 1) \ast b(x) \) in \( x \)-space to find \( b(x - 1) \), or in \( k \)-space to see the effect on the coefficients:

\[
\hat{b}(k) = \int_0^1 e^{-ikx} dx = \left[ e^{-ikx} \right]_{x=0}^{x=1} = \frac{1 - e^{-ik}}{ik}
\]

**Shifted box** \( e^{-ik} \left( \frac{1 - e^{-ik}}{ik} \right) \) agrees with \( \int_1^2 e^{-ikx} dx = \left[ e^{-ikx} \right]_{x=1}^{x=2} \).

12 Take the Laplace transform of these equations to find the transfer function \( G(s) \):

(a) \( Ay'' + By' + Cy = \delta(t) \)  (b) \( y' - 5y = \delta(t) \)  (c) \( 2y(t) - y(t - 1) = \delta(t) \)

**Solution** (a) \( As^2Y(s) + BsY(s) + Cy(s) = 1 \) gives the transfer function \( \frac{1}{As^2 + Bs + C} \).

**Solution** (b) \( sY(s) - 5Y(s) = 1 \) gives the transfer function \( Y(s) = \frac{1}{s - 5} \).
Solution (c)  \[ 2Y(s) - Y(s)e^{-s} = 1 \] gives the transfer function \[ Y(s) = \frac{1}{2 - e^{-s}} \]

13 Take the Laplace transform of \( y''' = \delta(t) \) to find \( Y(s) \). From the Transform Table in Section 8.5 find \( y(t) \). You will see \( y'' = 1 \) and \( y''' = 0 \). But \( y(t) = 0 \) for negative \( t \), so your \( y'' \) is actually a unit step function and your \( y''' \) is actually \( \delta(t) \).

Solution  \( y''' = \delta \) transforms to \( s^4 Y(s) - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0) = 1 \)

Assume zero initial values to get \( s^4 Y(s) = 1 \) and \( Y(s) = \frac{1}{s^4} \) and \( y^3 = \frac{t^3}{6} \).

This is also the solution to \( y''' = 0 \) with initial values \( y, y', y'', y''' = 0, 0, 0, 1 \).

14 Solve these equations by Laplace transform to find \( Y(s) \). Invert that transform with the Table in Section 8.5 to recognize \( y(t) \).

(a) \( y' - 6y = e^{-t}, y(0) = 2 \)  \hspace{1cm} (b) \( y'' + 9y = 1, y(0) = y'(0) = 0 \).

Solution (a) The transform of \( y' - 6y = e^{-t} \) with \( y(0) = 2 \) is
\[
sY(s) - 2 - 6Y(s) = \frac{1}{s + 1}
\]
\[
Y(s) = \frac{2}{s - 6} + \frac{1}{(s + 1)(s - 6)}
\]
\[
= \frac{2}{s - 6} + \frac{1}{7(s - 6)} - \frac{1}{7(s + 1)}
\]
\[
= \frac{15}{7(s - 6)} - \frac{7}{7(s + 1)}
\]
The inverse transform is \( y(t) = \frac{15}{7}e^{6t} - \frac{1}{7}e^{-t} \)

Solution (b) The transform of \( y'' + 9y = 1 \) with \( y(0) = y'(0) = 0 \) is
\[
s^2Y(s) + 9Y(s) = \frac{1}{s}
\]
\[
Y(s) = \frac{1}{s^2 + 9}
\]
\[
= \frac{1}{9s} - \frac{1}{18(3i + s)} - \frac{1}{18(3i + s)}
\]
The inverse transform is \( y(t) = \frac{1}{9} - \frac{1}{18}e^{3it} - \frac{1}{18}e^{-3it} = y_p + y_n. \)

15 Find the Laplace transform of the shifted step \( H(t - 3) \) that jumps from 0 to 1 at \( t = 3 \). Solve \( y' - ay = H(t - 3) \) with \( y(0) = 0 \) by finding the Laplace transform \( Y(s) \) and then its inverse transform \( y(t) \): one part for \( t < 3 \), second part for \( t \geq 3 \).

Solution The transform of \( H(t - 3) \) multiplies \( e^{-3t} \) by the transform \( \frac{1}{s} \) of \( H(t) \).

\[
y' - ay = H(t - 3) \quad y(0) = 0
\]
\[
sY(s) - aY(s) = \frac{e^{-3s}}{s - 3}
\]
\[
Y(s) = \frac{e^{-3s}}{s(s - 3)} = \frac{e^{-3s}}{3} \left( \frac{1}{s - 3} - \frac{1}{s} \right)
\]
The inverse transform \( y(t) \) is the shift of \( \frac{1}{3} \left( e^{-3t} - 1 \right) \): zero until \( t = 3 \).
16 Solve \( y' = 1 \) with \( y(0) = 4 \)—a trivial question. Then solve this problem the slow way by finding \( Y(s) \) and inverting that transform.

Solution The trivial solution is: \( y = t + 4 \). The transform method gives

\[
\begin{align*}
sY(s) - 4 &= \frac{1}{s} \\
Y(s) &= \frac{1}{s^2} + \frac{4}{s} \\
y(t) &= t + 4
\end{align*}
\]

17 The solution \( y(t) \) is the convolution of the input \( f(t) \) with what function \( g(t) \)?

(a) \( y' - ay = f(t) \) with \( y(0) = 3 \)

Solution (a) \( y' - ay = f(t) \) with \( y(0) = 3 \)

\[
\begin{align*}
sY(s) - 3 - aY(s) &= F(s) \\
Y(s) &= \frac{3 + F(s)}{s - a} \\
y(t) &= 3e^{-t} + f(t) * e^{-at}
\end{align*}
\]

(b) \( y' - \) (integral of \( y \)) = \( f(t) \).

Solution (b) The transform of \( y' - \) (integral of \( y \)) = \( f(t) \) is \( sY(s) - Y(s) = F(s) \), if \( y(0) = 0 \).

The inverse transform of \( \frac{1}{s - \alpha} \) is \( \cos(\alpha t) \).

Then \( Y(s) = \frac{F(s)}{s^2} \) is the transform of the convolution \( f(t) * \cos(\alpha t) \).

18 For \( y' - ay = f(t) \) with \( y(0) = 3 \), we could replace that initial value by adding \( 3\delta(t) \) to the forcing function \( f(t) \). Explain that sentence.

Solution For a first order equation, an initial condition \( y(0) \) is equivalent to adding \( y(0)\delta(t) \) to the equation and starting that new equation at zero.

19 What is \( \delta(t) * \delta(t) \)? What is \( \delta(t - 1) * \delta(t - 2) \)? What is \( \delta(t - 1) \) times \( \delta(t - 2) \)?

Solution \( \delta(t) * \delta(t) = \delta(t) \)

\( \delta(t - 1) * \delta(t - 2) = \delta(t - 3) \)

\( \delta(t - 1) \) times \( \delta(t - 2) \) equals the zero function.

20 By Laplace transform, solve \( y' = y \) with \( y(0) = 1 \) to find a very familiar \( y(t) \).

Solution \( y' = y \) \( y(0) = 1 \)

\[
\begin{align*}
sY(s) - 1 &= Y(s) \\
Y(s) &= \frac{1}{s - 1} \text{ gives } y(t) = e^t.
\end{align*}
\]
21 By Fourier transform as in (9), solve \(-y'' + y = \text{box function } b(x)\) on \(0 \leq x \leq 1\).

Solution The Fourier transform of \(-y'' + y = b(x)\) is
\[
(k^2 + 1) \hat{y}(k) = \hat{b}(k) = \int_0^1 e^{-ikx} \, dx = \frac{1 - e^{-ik}}{ik}.
\]

This transform must be inverted to find \(y(x)\). In reality I would solve separately on \(x \leq 0\) and \(0 \leq x \leq 1\) and \(x \geq 1\). Then matching at the breakpoints \(x = 0\) and \(x = 1\) determines the free constants in the separate solutions.

22 There is a big difference in the solutions to \(y'' + By' + Cy = f(x)\), between the cases \(B^2 < 4C\) and \(B^2 > 4C\). Solve \(y'' + y = \delta\) and \(y'' - y = \delta\) with \(y(\pm \infty) = 0\).

Solution (a) The delta function produces a unit jump in \(y'\) at \(x = 0\):
\[
y'' + y = 0 \text{ has } y = c_1 \cos x + c_2 \sin x \text{ for } x < 0, \quad y = C_1 \sin x \text{ for } x > 0.
\]

The jump in \(y'\) gives \(C_2 - c_2 = 1\). The condition on \(y(\pm \infty)\) does not apply to this first equation.

\[
y'' - y = 0 \text{ has } y = ce^x \text{ for } x < 0 \text{ and } y = Ce^{-x} \text{ for } x > 0; \text{ then } y(\pm \infty) = 0.
\]

Matching \(y\) at \(x = 0\) gives \(c = C\).

Jump in \(y'\) at \(x = 0\) gives \(-C - c = 1\) so \(c = C = -\frac{1}{2}\).

Solution \(y(x) = -\frac{1}{2}e^x\) for \(x \leq 0\) and \(y(x) = 1e^{-x}\) for \(x \geq 0\)

23 (Review) Why do the constant \(f(t) = 1\) and the unit step \(H(t)\) have the same Laplace transform \(1/s\)? Answer: Because the transform does not notice ______.

Solution The Laplace Transform does not notice any values of \(f(t)\) for \(t < 0\).