DIFFERENTIAL EQUATIONS
AND
LINEAR ALGEBRA

MANUAL FOR INSTRUCTORS

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Chapter 1. First Order Equations

Problem Set 1.1, page 3

1 Draw the graph of \( y = e^t \) by hand, for \(-1 \leq t \leq 1\). What is its slope \( dy/dt \) at \( t = 0 \)? Add the straight line graph of \( y = et \). Where do those two graphs cross?

Solution The derivative of \( e^t \) has slope 1 at \( t = 0 \). The graphs meet at \( t = 1 \) where their value is \( e \). They don’t actually “cross” because the line is tangent to the curve: both have slope \( y' = e \) at \( t = 1 \).

2 Draw the graph of \( y_1 = e^{2t} \) on top of \( y_2 = 2e^t \). Which function is larger at \( t = 0 \)? Which function is larger at \( t = 1 \)?

Solution From the graphs we see that at \( t = 0 \), the function \( 2e^t \) is larger whereas at \( t = 1 \), \( e^{2t} \) is larger. \( (e \times e) \) is larger than \( 2 \times e \).

3 What is the slope of \( y = e^{-t} \) at \( t = 0 \)? Find the slope \( dy/dt \) at \( t = 1 \).

Solution The slope of \( e^{-t} \) is \( -e^{-t} \). At \( t = 0 \) this is \(-1\). The slope at \( t = 1 \) is \(-e^{-1}\).

4 What “logarithm” do we use for the number \( t \) (the exponent) when \( e^t = 4 \)?

Solution We use the natural logarithm to find \( t \) from the equation \( e^t = 4 \). We get that \( t = \ln 4 \approx 1.386 \).

5 State the chain rule for the derivative \( dy/dt \) if \( y(t) = f(u(t)) \) (chain of \( f \) and \( u \)).

Solution The chain rule gives:

\[
\frac{dy}{dt} = \frac{df(u(t))}{du(t)} \frac{du(t)}{dt}
\]

6 The second derivative of \( e^t \) is again \( e^t \). So \( y = e^t \) solves \( d^2y/dt^2 = y \). A second order differential equation should have another solution, different from \( y = Ce^t \). What is that second solution?

Solution The second solution is \( y = e^{-t} \). The second derivative is \( -(e^{-t}) = -e^{-t} \).

7 Show that the nonlinear example \( dy/dt = y^2 \) is solved by \( y = C/(1 - Ct) \) for every constant \( C \). The choice \( C = 1 \) gave \( y = 1/(1 - t) \), starting from \( y(0) = 1 \).

Solution Given that \( y = C/(1 - Ct) \), we have:

\[
y^2 = C^2/(1 - Ct)^2 \\
\frac{dy}{dt} = C \cdot (-1) \cdot (-C)1/(1 - Ct)^2 = C^2/(1 - Ct)^2
\]

8 Why will the solution to \( dy/dt = y^2 \) grow faster than the solution to \( dy/dt = y \) (if we start them both from \( y = 1 \) at \( t = 0 \))? The first solution blows up at \( t = 1 \). The second solution \( e^t \) grows exponentially fast but it never blows up.

Solution The solution of the equation \( dy/dt = y^2 \) for \( y(0) = 1 \) is \( y = 1/(1 - t) \), while the solution to \( dy/dt = y \) for \( y(0) = 1 \) is \( y = e^t \). Notice that the first solution blows up at \( t = 1 \) while the second solution \( e^t \) grows exponentially fast but never blows up.
9 Find a solution to \( \frac{dy}{dt} = -y^2 \) starting from \( y(0) = 1 \). Integrate \( \frac{dy}{y^2} + (-dt) \) and \( \int y(0) \). From \( dz/dt = 1 \) you will know \( z(t) \) and \( y = 1/z \).

**Solution** The first method has \( \frac{dy}{y^2} = -dt \).

\[
\int y(0) = -t + \int_0^y \frac{du}{u^2}.
\]

\[
-\frac{1}{y} + \frac{1}{y(0)} = -t
\]

\[
-\frac{1}{y} = -t - 1
\]

\[
y = \frac{1}{1 + t}
\]

The approach using \( z = 1/y \) leads to \( dz/dt = 1 \) and \( z(0) = 1 \).

Then \( z(t) = 1 + t \) and \( y = 1/z \).

10 Which of these differential equations are linear (in \( y \))?

(a) \( y' + \sin y = t \) (b) \( y' = t^2(y - t) \) (c) \( y' + e^t y = t^{10} \).

**Solution** (a) Since this equation solves a \( \sin y \) term, it is not linear in \( y \).

(b) and (c) Since these equations have no nonlinear terms in \( y \), they are linear.

11 The product rule gives what derivative for \( e^t e^{-t} \)? This function is constant. At \( t = 0 \) this constant is 1. Then \( e^t e^{-t} = 1 \) for all \( t \).

**Solution** \((e^t e^{-t})' = e^t e^{-t} - e^t e^{-t} = 0\) so \( e^t e^{-t} \) is a constant \(1\).

12 \( dy/dt = y + 1 \) is not solved by \( y = e^t + t \). Substitute that \( y \) to show it fails. We can’t just add the solutions to \( y' = y \) and \( y' = 1 \). What number \( c \) makes \( y = e^t + c \) into a correct solution?

**Solution**

\[
\frac{d}{dt} = y + 1 \quad \frac{d(e^{t} + c)}{dt} = e^t + c + 1
\]

Wrong \( \frac{d(e^t + t)}{dt} = e^t + t + 1 \) Correct \( c = -1 \)

**Problem Set 1.3, page 15**

1 Set \( t = 2 \) in the infinite series for \( e^2 \). The sum must be \( e \) times \( e \), close to 7.39.

How many terms in the series to reach a sum of 7? How many terms to pass 7.3?

**Solution** The series for \( e^2 \) has \( t = 2 \): \( e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + 2^4 + \cdots \)

If we include five terms we get: \( e^2 \approx 1 + 2 + \frac{8}{6} + \frac{16}{24} = 7.0 \)

If we include seven terms we get: \( e^2 \approx 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} = 7.35556 \).
2 Starting from \( y(0) = 1 \), find the solution to \( \frac{dy}{dt} = y \) at time \( t = 1 \). Starting from that \( y(1) \), solve \( \frac{dy}{dt} = -y \) to time \( t = 2 \). Draw a rough graph of \( y(t) \) from \( t = 0 \) to \( t = 2 \). What does this say about \( e^{-t} \) times \( e \)?

Solution \( y = e^{t} \) up to \( t = 1 \), so that \( y(1) = e \). Then for \( t > 1 \) the equation \( \frac{dy}{dt} = -y \) has \( y = Ce^{-t} \). At \( t = 1 \), this becomes \( e = Ce^{-1} \) so that \( C = e^{2} \). The solution of \( \frac{dy}{dt} = -y \) up to \( t = 2 \) is \( y = e^{2-t} \). At \( t = 2 \) we have returned to \( y(2) = y(0) = 1 \). Then \((e^{-1})(e) = 1\).

3 Start with \( y(0) = $5000 \). If this grows by \( \frac{dy}{dt} = .02y \) until \( t = 5 \) and then jumps to \( a = .04 \) per year until \( t = 10 \), what is the account balance at \( t = 10 \)?

Solution

\[
\begin{align*}
5 \leq t \leq 10 : & \quad \frac{dy}{dt} = .04y \\
& \quad y(5) = 5000e^{.02t} \\
& \quad y(5) = 5000e^{.1} \text{ for } 5 \leq t \leq 10 \\
& \quad y(10) = 5000e^{.3}
\end{align*}
\]

4 Change Problem 3 to start with \$5000 growing at \( \frac{dy}{dt} = .04y \) for the first five years. Then drop to \( a = .02 \) per year until year \( t = 10 \). What is the account balance at \( t = 10 \)?

Solution

\[
\begin{align*}
\frac{dy}{dt} &= .04y \\
y & = C_{1}e^{.04t} \\
y(0) &= C_{1} = 5000 \\
y(t) &= 5000e^{.04t} \text{ for } 5 \leq t \leq 10 \\
y(10) &= 5000e^{.3}
\end{align*}
\]

Problems 5–8 are about \( y = e^{ax} \) and its infinite series.

5 Replace \( t \) by \( at \) in the exponential series to find \( e^{at} \):

\[
e^{at} = 1 + at + \frac{1}{2}(at)^{2} + \cdots + \frac{1}{n!}(at)^{n} + \cdots
\]

Take the derivative of every term*(keep five terms). Factor out \( a \) to show that the derivative of \( e^{at} \) equals \( ae^{at} \). At what time \( T \) does \( e^{at} \) reach 2?

Solution The derivative of this series is obtained by differentiating the terms individually:

\[
\frac{dy}{dt} = a + at + \cdots + \frac{1}{(n-1)!}a^{n}t^{n-1} + \cdots
\]

\[
= a \left(1 + at + \frac{1}{2}(at)^{2} + \cdots + \frac{1}{(n-1)!}a^{n-1}t^{n-1} + \cdots\right) = ae^{at}
\]

If \( e^{aT} = 2 \) then \( aT = \ln 2 \) and \( T = \frac{\ln 2}{a} \).

6 Start from \( y' = ay \). Take the derivative of that equation. Take the \( n^{th} \) derivative. Construct the Taylor series that matches all these derivatives at \( t = 0 \), starting from \( 1 + at + \frac{1}{2}(at)^{2} \). Confirm that this series for \( y(t) \) is exactly the exponential series for \( e^{at} \).

Solution The derivative of \( y' = ay \) is \( y'' = ay' = a^{2}y \). The next derivative is \( y''' = ay'' = a^{3}y \). When \( y(0) = 1 \), the derivatives at \( t = 0 \) are \( a, a^{2}, a^{3}, \ldots \) so the Taylor series is \( y(t) = 1 + at + \frac{1}{2}a^{2}t^{2} + \cdots = e^{at} \).
1.3. The Exponentials $e^t$ and $e^at$

7. At what times $t$ do these events happen?
   (a) $e^{at} = e$ at $t = 1/a$.
   (b) $e^{at} = e^2$ at $t = 2/a$.
   (c) $e^{a(t+2)} = e^{at}e^{2a}$.

Solution
(a) $e^{at} = e$ at $t = 1/a$.
(b) $e^{at} = e^2$ at $t = 2/a$.
(c) $e^{a(t+2)} = e^{at}e^{2a}$ at all $t$.

8. If you multiply the series for $e^{at}$ in Problem 5 by itself you should get the series for $e^{2at}$. Multiply the first 3 terms by the same 3 terms to see the first 3 terms in $e^{2at}$.

Solution
$$\begin{align*}
(1 + at + \frac{1}{2}a^2t^2)(1 + at + \frac{1}{2}a^2t^2) &= 1 + 2at + \left(1 + \frac{1}{2} + \frac{1}{2}\right)a^2t^2 + \cdots \\
&= 1 + 2at + \frac{1}{2}(2at)^2 + \cdots
\end{align*}$$

This agrees with $e^{2at} = 1 + 2at + \frac{1}{2}(2at)^2 + \cdots$.

9. (recommended) Find $y(t)$ if $dy/dt = ay$ and $y(T) = 1$ (instead of $y(0) = 1$).

Solution
\frac{dy}{dt} = ay \text{ gives } y(t) = Ce^{at}. \text{ When } Ce^{aT} = 1 \text{ at } t = T, \text{ this gives } C = e^{-aT} \text{ and } y(t) = e^{a(t-T)}.

10. (a) If $dy/dt = (\ln 2)y$, explain why $y(1) = 2y(0)$.
   (b) If $dy/dt = - (\ln 2)y$, how is $y(1)$ related to $y(0)$?

Solution
(a) \frac{dy}{dt} = (\ln 2)y \rightarrow y(t) = y(0)e^{t(\ln 2)} \rightarrow y(1) = y(0)e^{\ln 2} = 2y(0).

(b) \frac{dy}{dt} = - (\ln 2)y \rightarrow y(t) = y(0)e^{-t(\ln 2)} \rightarrow y(1) = y(0)e^{-\ln 2} = \frac{1}{2}y(0).

11. In a one-year investment of $y(0) = $100, suppose the interest rate jumps from 6% to 10% after six months. Does the equivalent rate for a whole year equal 8%, or more than 8%, or less than 8%?

Solution
We solve the equation in two steps, first from $t = 0$ to $t = 6$ months, and then from $t = 6$ months to $t = 12$ months.

\begin{align*}
y(t) &= y(0)e^{at} \\
y(0.5) &= y(0)e^{0.06 \times 0.5} = $100e^{0.03} \\
y(1) &= y(0.5)e^{at} = y(0.5)e^{0.1 \times 0.5} = $103.05e^{0.05} = $103.05 \\
&= $108.33
\end{align*}

If the money was invested for one year at 8% the amount at $t = 1$ would be:

$y(1) = $100e^{0.08 \times 1} = $108.33.

The equivalent rate for the whole year is indeed exactly 8%.

12. If you invest $y(0) = $100 at 4% interest compounded continuously, then $dy/dt = .04y$. Why do you have more than $104 at the end of the year?

Solution
The quantitative reason for why this is happening is obtained from solving the equation:
\begin{align*}
\frac{dy}{dt} &= 0.04y \rightarrow y(t) = y(0)e^{.04t} \\
y(1) &= 100e^{0.04} \approx $104.08.
\end{align*}

The intuitive reason is that the interest accumulates interest.
13 What linear differential equation \( dy/dt = a(t)y \) is satisfied by \( y(t) = e^{\cos t} \)?

**Solution** The chain rule for \( f(u(t)) \) has \( y(t) = f(u) = e^u \) and \( u(t) = \sin t \):

\[
\frac{dy}{dt} = \frac{df(u(t))}{dt} = \frac{df}{du} \frac{du}{dt} = e^u \cos t = y \cos t.
\]
Then \( a(t) = \cos(t) \).

14 If the interest rate is \( a = 0.1 \) per year in \( y' = ay \), how many years does it take for your investment to be multiplied by \( e \)? How many years to be multiplied by \( e^2 \)?

**Solution** If the interest rate is \( a = 0.1 \), then \( y(t) = y(0)e^{0.1t} \). For \( t = 10 \), the value is \( y(t) = y(0)e^{0.1\times10} \). For \( t = 20 \), the value is \( y(t) = y(0)e^{0.1\times20} \).

15 Write the first four terms in the series for \( y = e^t \). Check that \( dy/dt = 2ty \).

**Solution**

\[
y = e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots
\]

\[
\frac{dy}{dt} = 2t + 2t^2 + t^3 + \cdots = 2t\left(1 + t + \frac{t^2}{2} + \cdots\right) = 2te^t.
\]

16 Find the derivative of \( Y(t) = (1 + \frac{1}{n})^n \). If \( n \) is large, this \( dY/dt \) is close to \( Y' \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \!

17 (Key to future sections). Suppose the exponent in \( y = e^{u(t)} \) is \( u(t) = \) integral of \( a(t) \).

What equation \( dy/dt = \) ? does this solve? If \( u(0) = 0 \) what is the starting value \( y(0) \)?

**Solution** Differentiating \( y = e^{\int a(t)dt} \) with respect to \( t \) by the chain rule yields \( y' = a(t)e^{\int a(t)dt} \). Therefore \( dy/dt = a(t)y \). If \( u(0) = 0 \) we have \( y(0) = e^{u(0)} = 1 \).

18 The Taylor series comes from \( e^{d/dx}(f(x)) \), when you write out \( e^{d/dx} = 1 + d/dx + \frac{1}{2}(d/dx)^2 + \cdots \) as a sum of higher and higher derivatives. Applying the series to \( f(x) \) at \( x = 0 \) would give the value \( f + f' + \frac{1}{2}f'' + \cdots \) at \( x = 0 \).

The Taylor series says: This is equal to \( f(x) \) at \( x = \).

**Solution**

\[
f(1) = f(0) + f'(0) + \frac{1}{2}f''(0) + \cdots \text{ This is exactly}
\]

\[
f(1) = \left(1 + \frac{d}{dx} + \frac{1}{2} \left(\frac{d}{dx}\right)^2 + \cdots\right) f(x) \text{ at } x = 0.
\]

19 (Computer or calculator, 2.xx is close enough) Find the time \( t \) when \( e^t = 10 \).

The initial \( y(0) \) has increased by an order of magnitude—a factor of 10. The exact statement of the answer is \( t = \). At what time \( t \) does \( e^t \) reach 100?

**Solution** The exact time when \( e^t = 10 \) is \( t = \ln 10 \). This is \( t \approx 2.30 \) or \( 2.3026 \).

Then the time when \( e^t = 100 \) is \( T = \ln 100 = \ln 10^2 = 2 \ln 10 \approx 4.605 \).

Note that the time when \( e^t = \frac{1}{10} \) is \( t = - \ln 10 \) and not \( t = \frac{1}{\ln 10} \).

20 The most important curve in probability is the bell-shaped graph of \( e^{-t^2}/2 \).

With a calculator or computer find this function at \( t = -2, -1, 0, 1, 2 \). Sketch the graph of \( e^{-t^2}/2 \) from \( t = -\infty \) to \( t = \infty \). *It never goes below zero.*

**Solution** At \( t = 1 \) and \( t = -1 \), we have \( e^{-1^2}/2 = e^{-1/2} = 1/\sqrt{e} \approx .606 \)

At \( t = 2 \) and \( t = -2 \), we have \( e^{-2^2}/2 = e^{-2} \approx .13 \).
21 Explain why \( y_1 = e^{(a+b+c)t} \) is the same as \( y_2 = e^{at}e^{bt}e^{ct} \). They both start at \( y(0) = 1 \). They both solve what differential equation?

**Solution** The exponent rule is used twice to find $e^{(a+b+c)t} = e^{at}e^{bt}e^{ct} = e^{at}e^{bt}e^{ct}$.

This function must solve \( \frac{dy}{dt} = (a + b + c)y \). The product rule confirms this.

22 For \( y' = y \) with \( a = 1 \), Euler’s first step chooses \( Y_1 = (1 + \Delta t)Y_0 \). Backward Euler chooses \( Y_1 = Y_0/(1 - \Delta t) \). Explain why \( 1 + \Delta t \) is smaller than the exact \( e^{\Delta t} \).\( 1/(1 - \Delta t) \) is larger than \( e^{\Delta t} \). (Compare the series for \( 1/(1-x) \) with \( e^x \).)

**Solution** \( 1 + \Delta t \) is certainly smaller than \( e^{\Delta t} = 1 + \Delta t + \frac{1}{2}(\Delta t)^2 + \frac{1}{6}(\Delta t)^3 + \cdots \)

\[
\frac{1}{1 - \Delta t} = 1 + \Delta t + (\Delta t)^2 + (\Delta t)^3 + \cdots
\]

is larger than \( e^{\Delta t} \), because the coefficients drop below 1 in \( e^{\Delta t} \).

### Problem Set 1.4, page 27

1 All solutions to \( \frac{dy}{dt} = -y + 2 \) approach the steady state where \( \frac{dy}{dt} \) is zero and \( y = y_\infty = 2 \). That constant \( y = y_\infty \) is a particular solution \( y_p \).

Which \( y_n = C e^{-t} \) combines with this steady state \( y_p \) to start from \( y(0) = 4 \)?

This question chose \( y_p + y_n \) to be \( y_\infty \) + transient (decaying to zero).

**Solution** \( y_\infty = 2 = y_p \) at the steady state when \( \frac{dy}{dt} = 0 \). Then \( y_n = 2e^{-t} \) gives \( y = y_n + y_p = 2 + 2e^{-t} = 4 \) at \( t = 0 \).

2 For the same equation \( \frac{dy}{dt} = -y + 2 \), choose the null solution \( y_n \) that starts from \( y(0) = 4 \). Find the particular solution \( y_p \) that starts from \( y(0) = 0 \).

This splitting chooses \( y_n \) and \( y_p \) as \( e^{at}y(0) + \text{integral of } e^{a(t-T)}q \) in equation (4).

**Solution** For the same equation as 11.4.1, \( y_n = 4e^{-t} \) has the correct \( y(0) = 4 \). Now \( y_p \) must be \( 2 - 2e^{-t} \) to start at \( y_p(0) = 0 \). Of course \( y_n + y_p \) is still \( 2 + 2e^{-t} \).

3 The equation \( \frac{dy}{dt} = -2y + 8 \) also has two natural splittings \( y_S + y_T = y_N + y_P \):

1. Steady \( (y_S = y_\infty) + \text{Transient } (y_T \to 0) \). What are those parts if \( y(0) = 6 \)?
2. \( (y_N = -2y_P) \) from \( y_N(0) = 6 \) + \( (y'_P = -2y_P + 8 \) starting from \( y_P(0) = 0 \).

**Solution** 1. \( y_S = 4 \) (when \( \frac{dy}{dt} = 0 \): steady state) and \( y_T = 2e^{-2t} \).

2. \( y_N = 6e^{-2t} \) and \( y_P = 4 - 4e^{-2t} \) starts at \( y_P(0) = 0 \).

Again \( y_S + y_T = y_N + y_P \): two splittings of \( y \).

4 All null solutions to \( u - 2v = 0 \) have the form \( (u, v) = (c, \_\_\_) \).

One particular solution to \( u - 2v = 3 \) has the form \( (u, v) = (7, \_\_\_) \).

Every solution to \( u - 2v = 3 \) has the form \( (7, \_\_\_) + c(1, \_\_\_) \).

But also every solution has the form \( (3, \_\_\_) + C(1, \_\_\_) \) for \( C = c + 4 \).

**Solution** All null solutions to \( u - 2v = 0 \) have the form \( (u, v) = (c, \frac{1}{2}c) \).

One particular solution to \( u - 2v = 3 \) has the form \( (u, v) = (7, 2) \).

Every solution to \( u - 2v = 3 \) has the form \( (7, 2) + c(1, \frac{1}{2}) \).

But also every solution has the form \( (3, 0) + C(1, \frac{1}{2}) \). Here \( C = c + 4 \).
5 The equation \(dy/dt = 5\) with \(y(0) = 2\) is solved by \(y = \ldots\). A natural splitting \(y_n(t) = \ldots\) and \(y_p(t) = \ldots\) comes from \(y_n = e^{at}y(0)\) and \(y_p = \int e^{a(t-s)}5\,ds\).

This small example has \(a = 0\) (so \(ay\) is absent) and \(c = 0\) (the source is \(q = 5e^{0t}\)). When \(a = c\) we have “resonance.” A factor \(t\) will appear in the solution \(y\).

**Solution** \(dy/dt = 5\) with \(y(0) = 2\) is solved by \(y = 2 + 5t\). A natural splitting \(y_n(t) = 2\) and \(y_p(t) = 5t\) comes from \(y_n(0) = y(0)\) and \(y_p = \int e^{a(t-s)}5\,ds = 5t\) (since \(a = 0\)).

Starting with Problem 6, choose the very particular \(y_p\) that starts from \(y_p(0) = 0\).

6 For these equations starting at \(y(0) = 1\), find \(y_n(t)\) and \(y_p(t)\) and \(y(t) = y_n + y_p\).

(a) \(y' - 9y = 90\) \hspace{1cm} (b) \(y' + 9y = 90\)

**Solution** (a) Since the forcing function is \(a\) we use equation 6:

\[
y_n(t) = e^{9t}
y_p(t) = \frac{90}{e^{9t}}(e^{9t} - 1) = 10(e^{9t} - 1)
y(t) = y_n(t) + y_p(t) = e^{9t} + 10(e^{9t} - 1) = 11e^{9t} - 10.
\]

(b) We again use equation 6, noting that \(a = -9\). The steady state will be \(y_{\infty} = 10\).

\[
y_n(t) = e^{-9t}
y_p(t) = \frac{90}{-9}(e^{-9t} - 1)
y(t) = y_n(t) + y_p(t) = e^{-9t} - 10(e^{-9t} - 1) = 10 - 9e^{-9t}.
\]

7 Find a linear differential equation that produces \(y_n(t) = e^{2t}\) and \(y_p(t) = 5(e^{8t} - 1)\).

**Solution** \(y_n = e^{2t}\) needs \(a = 2\). Then \(y_p = 5(e^{8t} - 1)\) starts from \(y_p(0) = 0\), telling us that \(y(0) = y_n(0) = 1\). This \(y_p\) is a response to the forcing term \(e^{8t} + 1\). So the equation for \(y = e^{2t} + 5e^{8t} - 5\) must be \(\frac{dy}{dt} = 2y + (e^{8t} + 1)\). Substitute \(y:\)

\[
2e^{2t} + 40e^{8t} = 2e^{2t} + 10e^{8t} - 10 + (e^{8t} + 1).
\]

Comparing the two sides, \(C = 30\) and \(D = 10\). Harder than expected.

8 Find a resonant equation \((a = c)\) that produces \(y_n(t) = e^{2t}\) and \(y_p(t) = 3te^{2t}\).

**Solution** Clearly \(a = c = 2\). The equation must be \(dy/dt = 2y + Be^{2t}\). Substituting \(y = e^{2t} + 3te^{2t}\) gives \(2e^{2t} + 3e^{2t} + 6te^{2t} = 2e^{2t} + 3e^{2t} + Be^{2t}\) and then \(B = 3\).

9 \(y' = 3y + e^{3t}\) has \(y_n = e^{3t}y(0)\). Find the resonant \(y_p\) with \(y_p(0) = 0\).

**Solution** The resonant \(y_p\) has the form \(Cte^{3t}\) starting from \(y_p(0) = 0\). Substitute in the equation:

\[
\frac{dy}{dt} = 3y + e^{3t} \text{ is } Ce^{3t} + 3Cte^{3t} = 3Cte^{3t} + e^{3t} \text{ and then } C = 1.
\]

Problems 10–13 are about \(y' - ay = \text{constant source} q\).

10 Solve these linear equations in the form \(y = y_n + y_p\) with \(y_n = y(0)e^{at}\).

(a) \(y' - 4y = -8\) \hspace{1cm} (b) \(y' + 4y = 8\) \hspace{1cm} Which one has a steady state?

**Solution** (a) \(y' - 4y = -8\) has \(a = 4\) and \(y_p = 2\). But 2 is not a steady state at \(t = \infty\) because the solution \(y_n = y(0)e^{4t}\) is exploding.

(b) \(y' + 4y = 8\) has \(a = -4\) and again \(y_p = 2\). This 2 is a steady state because \(a < 0\) and \(y_n \to 0\).
11 Find a formula for \( y(t) \) with \( y(0) = 1 \) and draw its graph. What is \( y_\infty \)?

(a) \( y' + 2y = 6 \)  
(b) \( y' + 2y = -6 \)

**Solution**  
(a) \( y' + 2y = 6 \) has \( a = -2 \) and \( y_\infty = 3 \) and \( y = y(0)e^{-2t} + 3 \).  
(b) \( y' + 2y = -6 \) has \( a = -2 \) and \( y_\infty = -3 \) and \( y = y(0)e^{-2t} - 3 \).

12 Write the equations in Problem 11 as \( Y' = -2Y \) with \( Y = y - y_\infty \). What is \( Y(0) \)?

**Solution**  
With \( Y = y - y_\infty \) and \( Y(0) = y(0) - y_\infty \), the equations in 1.4.11 are \( Y' = -2Y \). The solutions are \( Y(t) = Y(0)e^{-2t} \) which is \( y(t) - y_\infty = (y(0) - y_\infty)e^{-2t} \) or \( y(t) = y(0)e^{-2t} + y_\infty (1 - e^{-2t}) \).

13 If a drip feeds \( q = 0.3 \) grams per minute into your arm, and your body eliminates the drug at the rate \( 6y \) grams per minute, what is the steady state concentration \( y_\infty \)? Then \( in = out \) and \( y_\infty \) is constant. Write a differential equation for \( Y = y - y_\infty \).

**Solution**  
The steady state has \( y_{in} = y_{out} \) or \( 0.3 = 6y_\infty \) or \( y_\infty = 0.05 \). The equation for \( Y = y - y_\infty \) is \( Y' = aY = -6Y \). The solution is \( Y(t) = Y(0)e^{-6t} \) or \( y(t) = y_\infty + (y(0) - y_\infty)e^{-6t} \).

**Problems 14–18 are about** \( y' - ay = \text{step function } H(t - T) \):

14 Why is \( y_\infty \) the same for \( y' + y = H(t - 2) \) and \( y' + y = H(t - 10) \)?

**Solution**  
Notice \( a = -1 \). The steady states are the same because the step functions \( H(t - 2) \) and \( H(t - 10) \) are the same after time \( t = 10 \).

15 Draw the ramp function that solves \( y' = H(t - T) \) with \( y(0) = 2 \).

**Solution**  
The solution is a ramp with \( y(t) = y(0) = 2 \) up to time \( T \) and then \( y(t) = 2 + t - T \) beyond time \( T \).

16 Find \( y_{in}(t) \) and \( y_{p}(t) \) as in equation (10), with step function inputs starting at \( T = 4 \).

(a) \( y' - 5y = 3H(t - 4) \)  
(b) \( y' + y = 7H(t - 4) \)  

**Solution**  
(a) \( y_{p}(t) = \frac{3}{5}(e^{5(t-4)} - 1) \) for \( t \geq 4 \) with no steady state.  
(b) \( y_{p}(t) = \frac{7}{2}(e^{-(t-4)} - 1) \) for \( t \geq 4 \) with \( a = -1 \) and \( y_\infty = 7 \).

17 Suppose the step function turns on at \( T = 4 \) and off at \( T = 6 \). Then \( q(t) = H(t - 4) - H(t - 6) \). Starting from \( y(0) = 0 \), solve \( y' + 2y = q(t) \). What is \( y_\infty \)?

**Solution**  
The solution has 3 parts. First \( y(t) = y(0) = 0 \) up to \( t = 4 \). Then \( H(t - 4) \) turns on and \( y(t) = \frac{1}{2}(e^{-2(t-4)} - 1) \). This reaches \( y(6) = -\frac{1}{2}(e^{-4} - 1) \) at time \( t = 6 \). After \( t = 6 \), the source is turned off and the solution decays to zero: \( y(t) = y(6)e^{-2(t-6)} \).

**Method 2:** We use the same steps as in equations (8) - (10), noting that \( y(0) = 0 \).

\[
(e^{2t}y)' = e^{2t}H(t - 4) - e^{2t}H(t - 6)
\]

\[
e^{2t}y(t) - e^{2t}y(0) = \int_{4}^{t} e^{2x} dx - \int_{6}^{t} e^{2x} dx
\]

\[
e^{2t}y(t) = -\frac{1}{2}(e^{2t} - e^{2t}H(t - 4) + \frac{1}{2}(e^{2t} - e^{2t})H(t - 4) - \frac{1}{2}(e^{2t} - e^{2t})H(t - 6)
\]

\[
y(t) = -\frac{1}{2}(e^{2t - 4} - 1)H(t - 4) + \frac{1}{2}(e^{2t - 4} - 1)H(t - 6)
\]
18 Suppose \( y' = H(t - 1) + H(t - 2) + H(t - 3) \), starting at \( y(0) = 0 \). Find \( y(t) \).

**Solution** We integrate both sides of the equation.

\[
\int_0^t y'(t) \, dt = \int_0^t (H(t - 1) + H(t - 2) + H(t - 3)) \, dt
\]

\[
y(t) - y(0) = R(t - 1) + R(t - 2) + R(t - 3)
\]

\[
y(t) = R(t - 1) + R(t - 2) + R(t - 3)
\]

\( R(t) \) is the unit ramp function = \( \max(0, t) \).

**Problems 19–25 are about delta functions and solutions to \( y' - ay = q \delta(t - T) \).**

19 For all \( t > 0 \) find these integrals \( a(t), b(t), c(t) \) of point sources and graph \( b(t) \):

- (a) \( \int_0^t \delta(T - 2) \, dT \)
- (b) \( \int_0^t (\delta(T - 2) - \delta(T - 3)) \, dT \)
- (c) \( \int_0^t \delta(T - 2)\delta(T - 3) \, dT \)

**Solution** For \( t < 2 \), the spike in \( \delta(t - 2) \) does not appear in the integral from 0 to \( t \):

\[
(a) \int_0^t \delta(T - 2) \, dT = \begin{cases} 0 & \text{if } t < 2 \\ 1 & \text{if } t \geq 2 \end{cases}
\]

The integral (b) equals 1 for \( 2 \leq t < 3 \). This is the difference \( H(t - 2) - H(t - 3) \). The integral (c) is zero because \( \delta(T - 2)\delta(T - 3) \) is everywhere zero.

20 Why are these answers reasonable? (They are all correct.)

- (a) \( \int_{-\infty}^\infty e^t \delta(t) \, dt = 1 \)
- (b) \( \int_{-\infty}^\infty (\delta(t))^2 \, dt = \infty \)
- (c) \( \int_{-\infty}^\infty e^T \delta(t - T) \, dT = e^t \)

**Solution** (a) The difference \( e^t \delta(t) - \delta(t) \) is everywhere zero (notice it is zero at \( t = 0 \)). So \( e^t \delta(t) \) and \( \delta(t) \) have the same integral (from \( -\infty \) to \( \infty \) that integral is 1). This reasoning can be made more precise.

(b) This is the difference between the step functions \( H(t - 2) \) and \( H(t - 3) \). So it equals 1 for \( 2 \leq t \leq 3 \) and otherwise zero.

(c) As in part (a), the difference between \( e^T \delta(t - T) \) and \( e^t \delta(t - T) \) is zero at \( t = T \) (and also zero at every other \( t \)). So

\[
\int_{-\infty}^\infty e^T \delta(t - T) \, dT = e^T \int_{-\infty}^\infty \delta(t - T) \, dT = e^t.
\]

21 The solution to \( y' = 2y + \delta(t - 3) \) jumps up by 1 at \( t = 3 \). Before and after \( t = 3 \), the delta function is zero and \( y \) grows like \( e^{2t} \). Draw the graph of \( y(t) \) when (a) \( y(0) = 0 \) and (b) \( y(0) = 1 \). Write formulas for \( y(t) \) before and after \( t = 3 \).
1.4. Four Particular Solutions

Solution (a) \( y(0) = 0 \) gives \( y(t) = 0 \) until \( t = 3 \). Then \( y(3) = 1 \) from the jump. After the jump we are solving \( y' = 2y \) and \( y \) grows exponentially from \( y(3) = 1 \). So \( y(t) = e^{2(t-3)} \).

(b) \( y(0) = 1 \) gives \( y(t) = e^{2t} \) until \( t = 3 \). The jump produces \( y(3) = e^6 + 1 \). Then exponential growth gives \( y(t) = e^{2(t-3)}(e^6 + 1) = e^{2t} + e^{2(t-3)} \). One part grows from \( t = 0 \), one part grows from \( t = 3 \) as before.

22 Solve these differential equations starting at \( y(0) = 2 \):
(a) \( y' - y = \delta(t-2) \)  (b) \( y' + y = \delta(t-2) \). (What is \( y_\infty \)?)

Solution (a) \( y' - y = \delta(t-2) \) starts with \( y(t) = y(0)e^t = 2e^t \) up to the jump at \( t = 2 \). The jump brings another term into \( y(t) = 2e^t + e^{t-2} \) for \( t \geq 2 \). Note the jump of \( e^{t-2} = 1 \) at \( t = 2 \).

(b) \( y' + y = \delta(t-2) \) starts with \( y(t) = y(0)e^{-t} = 2e^{-t} \) up to \( t = 2 \). The jump of \( 1 \) at \( t = 2 \) starts another exponential \( e^{-(t-2)} \) (decaying because \( a = -1 \)). Then \( y(t) = 2e^{-t} + e^{-(t-2)} \).

23 Solve \( dy/dt = H(t-1) + \delta(t-1) \) starting from \( y(0) = 0 \): jump and ramp.

Solution Nothing happens and \( y(t) = 0 \) until \( t = 1 \). Then \( H(t-1) \) starts a ramp in \( y(t) \) and there is a jump from \( \delta(t-1) \). So \( y(t) = \text{ramp} + \text{constant} = \max(0, t-1) + 1 \).

24 (My small favorite) What is the steady state \( y_\infty \) for \( y' = -y + \delta(t-1) + H(t-3) \)?

Solution \( dy/dt = 0 \) at the steady state \( y_{ss} \). Then \( -y + \delta(t-1) + H(t-3) = 0 \) and \( y_\infty = 1 \).

25 Which \( q \) and \( y(0) \) in \( y' - 3y = q(t) \) produce the step solution \( y(t) = H(t-1) \)?

Solution We simply substitute the particular solution \( y(t) = H(t-1) \) into the original differential equation with \( y(0) = 0 \):

\[ \delta(t-1) - 3H(t-1) = q(t) \]

Notice how \( \delta(t-1) \) in \( q(t) \) produces the jump \( H(t-1) \) in \( y \), and then \(-3H(t-1) \) in \( q(t) \) cancels the \(-3y \) and keeps \( dy/dt = 0 \) after \( t = 1 \).

Problems 26–31 are about exponential sources \( q(t) = Qe^{ct} \) and resonance.

26 Solve these equations \( y' - ay = Qe^{ct} \) as in (19), starting from \( y(0) = 2 \):
(a) \( y' - y = 8e^{3t} \)  (b) \( y' + y = 8e^{-3t} \). (What is \( y_\infty \)?)

Solution

(a) \( a = 1, c = 3 \) and \( y(0) = 2 \)
\[ y(t) = y(0)e^{at} + \frac{8e^{ct} - e^{at}}{c-a} \]
\[ y(t) = 2e^t + \frac{8e^{3t} - e^t}{3 - 1} \]
\[ y(t) = 2e^t + 4(e^{3t} - e^t) \]
\[ y(t) = 4e^{3t} - 2e^t \]
\( y \) goes to \( \infty \) as \( t \to \infty \)

(b) \( a = -1, c = -3 \) and \( y(0) = 2 \)
\[ y(t) = y(0)e^{at} + \frac{8e^{ct} - e^{at}}{c-a} \]
\[ y(t) = 2e^{-t} + \frac{8e^{-3t} - e^{-t}}{-3 - (-1)} \]
\[ y(t) = 2e^{-t} - 4(e^{-3t} - e^{-t}) \]
\[ y(t) = -4e^{-3t} + 2e^{-t} \]
\( y \) goes to \( 0 \) as \( t \to \infty \)
27 When \( c = 2.01 \) is very close to \( a = 2 \), solve \( y' - 2y = e^{ct} \) starting from \( y(0) = 1 \). By hand or by computer, draw the graph of \( y(t) \) near resonance.

**Solution** We substitute the values \( a = 2, c = 2.01 \) and \( y(0) = 1 \) into equation (18):

\[
y(t) = y(0)e^{at} + \frac{e^{ct} - e^{at}}{c - a} = y(0)e^{at} + \frac{e^{2t} - e^{2.01t}}{2.01 - 2} = y(0)e^{at} + \frac{e^{2t}}{2} - \frac{e^{2.01t}}{2.01} = 2e^{at} + \frac{e^{2t}}{2} - \frac{e^{2.01t}}{2.01}
\]

The graph of this function shows the “near resonance” when \( c \approx a \).

28 When \( c = 2 \) is exactly equal to \( a = 2 \), solve \( y' - 2y = e^{2t} \) starting from \( y(0) = 1 \). This is resonance as in equation (20). By hand or computer, draw the graph of \( y(t) \).

**Solution** We substitute \( a = 2, c = 2 \) (resonance) and \( y(0) = 1 \) into equation (19):

\[
y(t) = y(0)e^{at} + te^{at} = e^{2t} + te^{2t}
\]

29 Solve \( y' + 4y = 8e^{-4t} + 20 \) starting from \( y(0) = 0 \). What is \( y_{\infty} \)?

**Solution** We have \( a = -4, c = -4 \) and \( y(0) = 0 \). Equation (19) with resonance leads to \( 8te^{-4t} \). The constant source 20 leads to \( 20(e^{-4t} - 1) \). By linearity \( y(t) = 8te^{-4t} + 20(e^{-4t} - 1) \). The steady state is \( y_{\infty} = -20 \).

30 The solution to \( y' - ay = e^{ct} \) didn’t come from the main formula (4), but it could. Integrate \( e^{-ax}e^{ct} \) in (4) to reach the very particular solution \( (e^{ct} - e^{at})/(c - a) \).
1.4. Four Particular Solutions

Solution

\[ y(t) = e^{at}y(0) + e^{at} \int_{0}^{t} e^{-aT} q(T) dT \]

\[ = e^{at}y(0) + e^{at} \int_{0}^{t} e^{-aT} e^{cT} dT \]

\[ = e^{at}y(0) + e^{at} \int_{0}^{t} e^{(c-a)T} dT \]

\[ = e^{at}y(0) + e^{at} \left( \frac{e^{(c-a)t} - e^{0}}{c - a} \right) \]

\[ = e^{at}y(0) + \frac{e^{ct} - e^{at}}{c - a} = y_n + y_p \]

31 The easiest possible equation \( y' = 1 \) has resonance! The solution \( y = t \) shows the factor \( t \). What number is the growth rate \( a \) and also the exponent \( c \) in the source?

Solution  The growth rate in \( y' = 1 \) or \( dy/dt = e^{0t} \) is \( a = 0 \). The source is \( e^{ct} \) with \( c = 0 \). Resonance \( a = c \). The resonant solution \( y(t) = te^{at} \) is certainly correct for the equation \( dy/dt = 1 \).

32 Suppose you know two solutions \( y_1 \) and \( y_2 \) to the equation \( y' - a(t)y = q(t) \).

(a) Find a null solution to \( y' - a(t)y = 0 \).

(b) Find all null solutions \( y_n \). Find all particular solutions \( y_p \).

Solution  (a) \( y = y_1 - y_2 \) will be a null solution by linearity.

(b) \( y = C(y_1 - y_2) \) will give all null solutions. Then \( y = C(y_1 - y_2) + y_2 \) will give all particular solutions. (Also \( y = c(y_1 - y_2) + y_2 \) will also give all particular solutions.)

33 Turn back to the first page of this Section 1.4. Without looking, can you write down a solution to \( y' - a(t)y = q(t) \) for all four source functions \( q, H(t), \delta(t), e^{ct} \)?

Solution  Equations (5), (7), (14), (19).

34 Three of those sources in Problem 33 are actually the same, if you choose the right values for \( q \) and \( c \) and \( y(0) \). What are those values?

Solution  The sources \( q = 1 \) and \( q = H(t) \) and \( q = e^{0t} \) are all the same for \( t \geq 0 \).

35 What differential equations \( y' = ay + q(t) \) would be solved by \( y_1(t) \) and \( y_2(t) \)? Jumps, ramps, corners—maybe harder than expected (math.mit.edu/dela/Pset1.4).
Chapter 1. First Order Equations

Solution (a) \( \frac{dy_1}{dt} = 1 - \delta(t - 1) - \delta(t - 2) \) with \( a = 0 \).

(b) \( \frac{dy_2}{dt} = y_2 + 1 \) up to \( t = 1 \). Add in \(-2e \delta(t - 1)\) to drop the slope from \( e \) to \(-e\) at \( t = 1 \). After \( t = 1 \) we need \( \frac{dy_2}{dt} = -y_2 - 1 \) to keep \( y_2 = e^{2-t} - 1 \).

Problem Set 1.5, page 37

Problems 1-6 are about the sinusoidal identity (9). It is stated again in Problem 1.

1. These steps lead again to the sinusoidal identity. This approach doesn’t start with the usual formula \( \cos(\omega t - \phi) = \cos \omega t \cos \phi + \sin \omega t \sin \phi \) from trigonometry. The identity says:

   If \( A + iB = Re^{i\phi} \) then \( A \cos \omega t + B \sin \omega t = R \cos(\omega t - \phi) \).

Here are the four steps to find that real part of \( Re^{i(\omega t - \phi)} \). Explain Step 3 where \( Re^{i\phi} \) equals \( A + iB \):

\[
R \cos (\omega t - \phi) = Re \left[ Re^{i(\omega t - \phi)} \right] = Re \left[ e^{i\omega t} \right] = (what \ is \ R \ e^{-i\phi}?)
\]

\[
= Re \left[ \left( \cos \omega t + i \sin \omega t \right) \left( A - iB \right) \right] = A \cos \omega t + B \sin \omega t.
\]

Solution The key point is that if \( A + iB = Re^{i\phi} \) then \( A - iB = Re^{-i\phi} \) (the complex conjugate).

2. To express \( \sin 5t + \cos 5t \) as \( R \cos(\omega t - \phi) \), what are \( R \) and \( \phi \)?

Solution The sinusoidal identity has \( A = 1, B = 1, \) and \( \omega = 5 \). Therefore:

\[
R^2 = A^2 + B^2 = 2 \rightarrow R = \sqrt{2} \text{ and } \tan \phi = \frac{1}{1} \rightarrow \phi = \frac{\pi}{4} \text{. Answer } \sqrt{2} \cos \left( 5t - \frac{\pi}{4} \right).
\]

3. To express \( 6 \cos 2t + 8 \sin 2t \) as \( R \cos(2t - \phi) \), what are \( R \) and \( \tan \phi \) and \( \phi \)?

Solution Use the Sinusoidal Identity with \( A = 6, B = 8 \) and \( \omega = 2 \).

\[
R^2 = A^2 + B^2 = 6^2 + 8^2 = 100 \text{ and } R = 10
\]

\[
\tan \phi = \frac{B}{A} = \frac{8}{6} = \frac{4}{3} \text{ and } \phi \text{ is in the positive quadrant } 0 \text{ to } \frac{\pi}{2} \text{ (not } \pi \text{ to } \frac{3\pi}{2})
\]

\[
6 \cos(2t) + 8 \sin(2t) = 10 \cos \left( 2t - \arctan \left( \frac{4}{3} \right) \right).
\]

4. Integrate \( \cos \omega t \) to find \( (\sin \omega t)/\omega \) in this complex way.

(i) \( \frac{dy_{\text{real}}}{dt} = \cos \omega t \) is the real part of \( \frac{dy_{\text{complex}}}{dt} = e^{i\omega t} \).

(ii) Take the real part of the complex solution.

Solution (i) The complex equation \( y' = e^{i\omega t} \) leads to \( y = \frac{e^{i\omega t}}{i\omega} \).

(ii) Take the real part of that solution (since the real part of the right side is \( \cos \omega t \)).

\[
\text{Re} \frac{e^{i\omega t}}{i\omega} = \text{Re} \left[ \frac{\cos \omega t}{i\omega} + \frac{\sin \omega t}{\omega} \right] = \frac{\sin \omega t}{\omega}.
\]
1.5. Real and Complex Sinusoids

5 The sinusoidal identity for \( A = 0 \) and \( B = -1 \) says that \(- \sin \omega t = R \cos(\omega t - \phi)\). Find \( R \) and \( \phi \).

**Solution** \( R^2 = A^2 + B^2 = 0^2 + 1^2 = 1 \Rightarrow R = 1 \)
\[
\tan \phi = \frac{1}{0} = \infty \Rightarrow \phi = \frac{\pi}{2} \text{ or } \phi = \frac{3\pi}{2} : \text{ Here it is } \frac{3\pi}{2} \text{, since } A + iB = -i
\]
Therefore we have

**SOLUTION:** \(- \sin \omega t = \cos(\omega t - \frac{3\pi}{2})

**CHECK:** \( t = 0 \) gives \( 0 = 0, \omega t = \frac{\pi}{2} \) gives \(- 1 = -1\).

6 Why is the sinusoidal identity useless for the source \( q(t) = \cos t + \sin 2t \)?

**Solution** The sinusoidal identity needs the same \( \omega \) in all terms. But the first term has \( \omega = 1 \) while the second term has \( \omega = 2 \).

7 Write \( 2 + 3i \) as \( re^{i\phi} \), so that \( \frac{1}{2 + 3i} = \frac{1}{2} e^{-i\phi} \). Then write \( y = e^{i\omega t}/(2 + 3i) \) in polar form. Then find the real and imaginary parts of \( y \). And also find those real and imaginary parts directly from \( (2 - 3i)e^{i\omega t}/(2 - 3i)(2 + 3i) \).

**Solution**
\[
r = \sqrt{2^2 + 3^2} = \sqrt{13} \text{ and } \phi = \arctan(3/2)
\]
\[2 + 3i = \sqrt{13} e^{i \arctan(3/2)} \]
\[y = e^{i\omega t}/(2 + 3i) = \sqrt{13} e^{i \arctan(3/2) + i\omega t}
\]
Writing this in cartesian (rectangular) form gives

real part = \( \sqrt{13} \cos(\arctan(3/2) + \omega t) = 2 \cos(\omega t) - 3 \sin(\omega t) \)

imag part = \( \sqrt{13} \sin(\arctan(3/2) + \omega t) = 3 \cos(\omega t) + 2 \sin(\omega t) \)

We can also find the real and imaginary parts from:
\[
\frac{(2 - 3i)e^{i\omega t}}{(2 - 3i)(2 + 3i)} = \frac{2 - 3i}{13} e^{i\omega t} = \frac{2 - 3i}{13} (\cos(\omega t) + i \sin(\omega t)).
\]

8 Write these functions \( A \cos \omega t + B \sin \omega t \) in the form \( R \cos(\omega t - \phi) \): Right triangle with sides \( A, B, R \) and angle \( \phi \).

1. \( \cos 3t - \sin 3t \)
2. \( \sqrt{3} \cos \pi t - \sin \pi t \)
3. \( 3 \cos(t - \phi) + 4 \sin(t - \phi) \)

**Solution**
\[
1. \cos 3t - \sin 3t = \sqrt{2} \cos(3t - \frac{\pi}{4}) = \sqrt{2} \cos(\pi t + \frac{\pi}{4})
\]
Check \( t = 0 : 1 = \sqrt{2} \cos(- \frac{\pi}{4}) = \sqrt{2} \cos(\frac{\pi}{4}) \).

2. \( \sqrt{3} \cos \pi t - \sin \pi t = 2 \cos(\pi t + \frac{\pi}{6}) \).

Check: \( (\sqrt{3})^2 + (-1)^2 = 2^2 \) \( \text{At } t = 0 : \sqrt{3} = 2 \cos 30^\circ \).

3. \( 3 \cos(t - \phi) + 4 \sin(t - \phi) = 5 \cos(t - \phi - \tan^{-1} \frac{3}{4}) \).

Problems 9-15 solve real equations using the real formula (3) for \( M \) and \( N \).

9 Solve \( dy/dt = 2y + 3 \cos t + 4 \sin t \) after recognizing \( a \) and \( \omega \). Null solutions \( Ce^{2t} \).

**Solution** \( \frac{dy}{dt} = 2y + 3 \cos t + 4 \sin t = 2y + 5 \cos(t - \phi) \) with \( \tan \phi = \frac{1}{3} \).

Method 1: Look for \( y = M \cos t + N \sin t \).

Method 2: Solve \( \frac{dy}{dt} = 2Y + 5e^{i(t-\phi)} \) and then \( y \) is real part of \( Y \).
\[
Y = \frac{5}{i-2} e^{i(t-\phi)} = \frac{5}{5}(-i-2)e^{i(t-\phi)} \text{ and } y = -2 \cos(t - \phi) + \sin(t - \phi).
\]
10 Find a particular solution to \( \frac{dy}{dt} = -y - \cos 2t \).

**Solution**  Substitute \( y = M \cos t + N \sin t \) into the equation to find \( M \) and \( N \).

\[-M \sin t + N \cos t = -M \cos t - N \sin t - \cos 2t \]

Match coefficients of \( \cos t \) and \( \sin t \) separately to find \( M \) and \( N \).

\[N = -M - 1 \quad \text{and} \quad -M = -N \quad \text{give} \quad M = N = -\frac{1}{2} \]

Note: This is called the “method of undetermined coefficients” in Section 2.6.

11 What equation \( y' - ay = A \cos \omega t + B \sin \omega t \) is solved by \( y = 3 \cos 2t + 4 \sin 2t \)?

**Solution**  Clearly \( \omega = 2 \). Substitute \( y \) into the equation:

\[-6 \sin 2t + 8 \cos 2t - 3a \cos 2t - 4a \sin 2t = A \cos 2t + B \sin 2t.\]

Match separately the coefficients of \( \cos 2t \) and \( \sin 2t \):

\[A = 8 - 3a \quad \text{and} \quad B = -6 - 4a \]

12 The particular solution to \( y' = y + \cos t \) in Section 4 is \( y_p = \int e^{at} \cos \omega s \, ds \). Look this up or integrate by parts, from \( s = 0 \) to \( t \). Compare this \( y_p \) to formula (3).

**Solution**  That integral goes from \( 0 \) to \( t \), and it leads to \( y_p = \frac{1}{2}(\sin t - \cos t + e^t) \)

If we use formula (3) with \( a = 4, \omega = 3, A = B = 1 \) we get

\[M = \frac{4 + 3}{9 + 16} = -\frac{7}{25} \quad N = \frac{-A^2}{\omega^2 + a^2} = -\frac{1}{25} \]

This solution \( y = M \cos t + N \sin t = -\cos t + \sin t \) is a different particular solution (not starting from \( y(0) = 0 \)). The difference is a null solution \( \frac{1}{2} e^t \).

13 Find a solution \( y = M \cos \omega t + N \sin \omega t \) to \( y' - 4y = \cos 3t + \sin 3t \).

**Solution**  Formula (3) with \( a = 4, \omega = 3, A = B = 1 \) gives

\[M = -\frac{4 + 3}{9 + 16} = -\frac{7}{25} \quad N = \frac{3 - 4}{9 + 16} = -\frac{1}{25} \]

14 Find the solution to \( y' - ay = A \cos \omega t + B \sin \omega t \) starting from \( y(0) = 0 \).

**Solution**  One particular solution \( M \cos \omega t + N \sin \omega t \) comes from formula (3). But this starts from \( y_p(0) = M \). So subtract off the null solution \( y_n = Me^{at} \) to get the very particular solution \( y_{vp} = y_p - y_n \) that starts from \( y_{vp}(0) = 0 \).

15 If \( a = 0 \) show that \( M \) and \( N \) in equation (3) still solve \( y' = A \cos \omega t + B \sin \omega t \).

**Solution**  Formula (3) still applies with \( a = 0 \) and it gives

\[M = -\frac{\omega B}{\omega^2} \quad N = \frac{\omega A}{\omega^2} \quad y = -\frac{B}{\omega} \cos \omega t + \frac{A}{\omega} \sin \omega t. \]

This is the correct integral of \( A \cos \omega t + B \sin \omega t \) in the differential equation.
1.5. Real and Complex Sinusoids

Problems 16-20 solve the complex equation $y' - ay = Re^{i(\omega t - \phi)}$.

16 Write down complex solutions $y_p = Y e^{i\omega t}$ to these three equations:
   (a) $y' - 3y = 5e^{2it}$
   (b) $y' = Re^{i(\omega t - \phi)}$
   (c) $y' = 2y - e^{it}$

Solution  
   (a) $y' - 3y = 5e^{2it}$ has $i\omega Y e^{i\omega t} - 3Y e^{i\omega t} = 5e^{2it}$.
   So $\omega = 2$ and $Y = \frac{5}{2i - 3}$.

   (b) $y' = Re^{i(\omega t - \phi)}$ has $i\omega Y e^{i\omega t} = Re^{i(\omega t - \phi)}$. So $Y = \frac{R}{i\omega} e^{-i\phi}$ and the solution is
   $y = Ye^{i\omega t} = Re^{i(\omega t - \phi)}$.

   (c) $y' = 2y - e^{it}$ has $\omega = 1$ and $iY e^{it} = 2Y e^{it} - e^{-it}$.
   Then $Y = \frac{1}{2 - i} = \frac{2 + i}{5}$ and $y = Ye^{it}$.

17 Find complex solutions $z_p = Ze^{i\omega t}$ to these complex equations:
   (a) $z' + 4z = e^{8it}$
   (b) $z' + 4iz = e^{8it}$
   (c) $z' + 4iz = e^{8it}$

Solution  
   (a) $z' + 4z = e^{8it}$ has $z = Ze^{8it}$ with $8iZ + 4Z = 1$ and $Z = \frac{1}{4 + 8i} = \frac{1 - 2i}{20} = \frac{1}{20}(1 - 2i)$.

   (b) $z' + 4iz = e^{8it}$ is like part (a) but 4 changes to $4i$. Then $Z = \frac{1}{4 + 8i} = \frac{1}{12} = -\frac{1}{12}$.

   (c) $z' + 4iz = e^{8it}$ has $z = Ze^{8it}$. Then $8Ze^{8t} + 4iZe^{8t}$ gives $Z = \frac{1}{8 + 4i} = \frac{8 - 4i}{8^2 + 4^2}$.

18 Start with the real equation $y' - ay = R \cos(\omega t - \phi)$. Change to the complex equation $z' - az = Re^{i(\omega t - \phi)}$. Solve for $z(t)$. Then take its real part $y_p = Re z$.

   Solution  
   Put $z = Ze^{i(\omega t - \phi)}$ in the complex equation to find $Z$:
   $i\omega Z - aZ = R$ gives $Z = \frac{R}{-a + i\omega} = \frac{R(-a - i\omega)}{a^2 + \omega^2}$.

   The real part of $z = Z(\cos(\omega t - \phi) + i\sin(\omega t - \phi))$ is
   $\frac{R}{a^2 + \omega^2}(-a \cos(\omega t - \phi) + \omega \sin(\omega t - \phi))$.

19 What is the initial value $y_p(0)$ of the particular solution $y_p$ from Problem 18? If the desired initial value is $y(0)$, how much of the null solution $y_n = Ce^{\alpha t}$ would you add to $y_p$?

   Solution  
   That solution to 18 starts from $y_p(0) = \frac{R}{a^2 + \omega^2}(-a \cos(\phi) + \omega \sin(\phi))$ at $t = 0$. So subtract that number times $e^{\alpha t}$ to get the very particular solution that starts from $y_{p,0}(0) = 0$.

20 Find the real solution to $y' - 2y = \cos \omega t$ starting from $y(0) = 0$, in three steps: Solve the complex equation $z' - 2z = e^{it}$, take $y_p = Re z$, and add the null solution $y_n = Ce^{2it}$ with the right $C$.

   Solution  
   Step 1. $z' - 2z = e^{it}$ is solved by $z = Ze^{it}$ with $i\omega Z - 2Z = 1$ and $Z = \frac{1}{-2 + i\omega} = \frac{2 - i\omega}{4 + \omega^2}$.

   Step 2. The real part of $Ze^{it}$ is $y_p = \frac{1}{4 + \omega^2}(-2 \cos \omega t + \omega \sin \omega t)$.

   Step 3. $y_p(0) = \frac{2}{4 + \omega^2}$ so $y_{vp} = y_p + \frac{2}{4 + \omega^2}e^{2it}$ includes the right $y_n = Ce^{2it}$ for $y_{vp}(0) = 0$. 

Problems 21-27 solve real equations by making them complex. First a note on $\alpha$.

Example 4 was $y' - y = \cos t - \sin t$, with growth rate $a = 1$ and frequency $\omega = 1$. The magnitude of $i\omega - a = \sqrt{2}$ and the polar angle has $\tan \alpha = -\omega/a = -1$. Notice: Both $\alpha = 3\pi/4$ and $\alpha = -\pi/4$ have that tangent! How to choose the correct angle $\alpha$?

The complex number $i\omega - a = i - 1$ is in the second quadrant. Its angle is $\alpha = 3\pi/4$.
We had to look at the actual number and not just the tangent of its angle.

21 Find $r$ and $\alpha$ to write each $i\omega - a$ as $re^{i\alpha}$. Then write $1/re^{i\alpha}$ as $Ge^{-i\alpha}$.
(a) $\sqrt{3}i + 1$  
Solution (a) $\sqrt{3}i + 1$ is in the first quadrant (positive quarter $0 \leq \theta \leq \pi/2$) of the complex plane. The angle with tangent $\sqrt{3}/1$ is $60^\circ = \pi/3$. The magnitude of $\sqrt{3}i + 1$ is $r = 2$. Then $\sqrt{3}i + 1 = 2e^{i\pi/3}$.
(b) $\sqrt{3}i - 1$ is in the second quadrant $\pi/2 \leq \theta \leq \pi$. The tangent is $-\sqrt{3}$, the angle is $\theta = 2\pi/3$, the number is $2e^{2\pi i/3}$.
(c) $i - \sqrt{3}$ is also in the second quadrant (left from zero and up). Now the tangent is $-1/\sqrt{3}$, the angle is $\theta = 150^\circ = 5\pi/6$. The magnitude is still 2, the number is $2e^{5\pi i/6}$.

22 Use $G$ and $\alpha$ from Problem 21 to solve (a)-(b)-(c). Then take the real part of each equation and the real part of each solution.
(a) $y' + y = e^{\sqrt{3}t}$  
Solution (a) $y' + y = e^{i\sqrt{3}t}$ is solved by $y = Ye^{i\sqrt{3}t}$ when $i\sqrt{3}Y + Y = 1$. Then $Y = \frac{1}{\sqrt{3}i+1} = \frac{1}{2}e^{-i\pi/3}$ from Problem 21(a). The real part $y_{\text{real}} = \frac{1}{2}\cos(\sqrt{3}t - \pi/3)$ of $Ye^{i\sqrt{3}t}$ solves the real equation $y'_{\text{real}} + y_{\text{real}} = \cos(\sqrt{3}t)$.
(b) $y' - y = e^{i\sqrt{3}t}$ is solved by $y = Ye^{i\sqrt{3}t}$ when $i\sqrt{3}Y - Y = 1$. Then $Y = \frac{1}{2}e^{-2\pi i/3}$ from Problem 21(b). The real part $y_{\text{real}} = \frac{1}{2}\cos(\sqrt{3}t - 2\pi/3)$ solves the real equation $y'_{\text{real}} - y_{\text{real}} = \cos(\sqrt{3}t)$.
(c) $y' + \sqrt{3}y = e^{it}$ is solved by $y = Ye^{it}$ when $iY + \sqrt{3}Y = 1$. Then $Y = \frac{1}{2}e^{-5\pi i/6}$ from Problem 21(c). The real part $y_{\text{real}} = \frac{1}{2}\cos(t - 5\pi/6)$ of $Ye^{it}$ solves $y'_{\text{real}} + \sqrt{3}y_{\text{real}} = \cos t$.

23 Solve $y' - y = \cos \omega t + \sin \omega t$ in three steps: real to complex, solve complex, take real part. This is an important example.

Solution Note: I intended to choose $\omega = 1$. Then $y' - y = \cos t + \sin t$ has the simple solution $y = -\sin t$. I will apply the 3 steps to this case and then to the harder problem for any $\omega$.

(1) Find $R$ and $\phi$ in the sinusoidal identity to write $\cos \omega t + \sin \omega t$ as the real part of $Re^{i(\omega t - \phi)}$. This is easy for any $\omega$.

\[
\tan \phi = \frac{1}{1} \quad \text{so} \quad \phi = \frac{\pi}{4} \quad \cos \omega t + \sin \omega t = \sqrt{2} \cos \left( \omega t - \frac{\pi}{4} \right)
\]

(2) Solve $y' - y = e^{i\omega t}$ by $y = Ge^{-i\alpha}e^{i\omega t}$. Multiply by $Re^{-i\phi}$ to solve
1.5. Real and Complex Sinusoids

\[ \omega = 1 \quad y' - y = e^{it} \] has \( y = Y e^{it} \) with \( iY - Y = 1 \). Then \( Y = \frac{1}{i - 1} = \frac{1}{\sqrt{2}} e^{3\pi i/4} = G e^{-i\alpha} \).

\[ z = (\sqrt{2} e^{i(t - \pi/4)}) \left( \frac{1}{\sqrt{2}} e^{3\pi i/4} \right) = e^{i} e^{\pi i/2} = i e^{it} \]. The real part of \( z \) is \( y = -\sin t \).

Any \( \omega \) \( y' - y = e^{i\omega t} \) leads to \( i\omega Y - Y = 1 \) and \( Y = \frac{1}{i\omega - 1} = \frac{1}{\sqrt{1 + \omega^2}} e^{-i\alpha} \)

with \( \tan \alpha = \omega \). Then \( z(t) = \left( \frac{1}{1 + \omega^2} e^{-i\alpha} \right) \left( \sqrt{2} e^{i(t - \pi/4)} \right) \).

(3) Take the real part \( y(t) = \Re z(t) \). Check that \( y' - y = \cos \omega t + \sin \omega t \).

\[ y(t) = \Re z(t) = \frac{\sqrt{2}}{1 + \omega^2} \cos(\omega t - \alpha - \frac{\pi}{4}) \]. Now we need \( \tan \alpha = \omega, \cos \alpha = \frac{1}{\sqrt{1 + \omega^2}}, \sin \alpha = \frac{\omega}{\sqrt{1 + \omega^2}} \). Finally \( y = \frac{\sqrt{2}}{1 + \omega^2} [\cos(\omega t - \frac{\pi}{4}) \cos \alpha + \sin(\omega t - \frac{\pi}{4}) \sin \alpha] \).

24 Solve \( y' - \sqrt{3} y = \cos t + \sin t \) by the same three steps with \( a = \sqrt{3} \) and \( \omega = 1 \).

\[ \text{Solution} \quad (1) \cos t + \sin t = \sqrt{2} \cos(t - \frac{\pi}{4}) \).

(2) \( y = Y e^{it} \) with \( iY - \sqrt{3} Y = 1 \) and \( Y = \frac{1}{1 - \sqrt{3}} = \frac{1}{2} e^{-5\pi i/6} \) from 1.5.21(c).

Then \( z(t) = \left( \frac{\sqrt{2} e^{i(t - \pi/4)}}{1 - e^{5\pi i/6}} \right) \).

(3) The real part of \( z(t) \) is \( y(t) = \frac{1}{\sqrt{2}} \cos(t - \frac{13\pi}{12}) \).

25 (Challenge) Solve \( y' - ay = A \cos \omega t + B \sin \omega t \) in two ways. First, find \( R \) and \( \phi \) on the right side and \( G \) and \( \alpha \) on the left. Show that the final real solution \( RG \cos(\omega t - \phi - \alpha) \) agrees with \( M \cos \omega t + N \sin \omega t \) in equation (3).

\[ \text{Solution} \quad \text{The first way has} \quad R = \sqrt{A^2 + B^2} \text{and} \quad \tan \phi = B/A \text{from the sinusoidal identity. On the left side} \quad 1/(i\omega - a) = Ge^{-i\alpha}, \text{from equation (8) with} \quad G = 1/\sqrt{\omega^2 + a^2} \text{and} \quad \tan \alpha = -\omega/a. \text{Combining, the real solution is} \quad y = RG \cos(\omega t - \phi - \alpha). \]

This agrees with \( y = M \cos \omega t + N \sin \omega t \) (equation (3) gives \( M \) and \( N \)).

26 We don’t have resonance for \( y' - ay = Re^{i\omega t} \) when \( a \) and \( \omega \neq 0 \) are real. Why not? (Resonance appears when \( y_n = Ce^{at} \) and \( y_p = Y e^{it} \) share the exponent \( a = c \).)

\[ \text{Solution} \quad \text{Resonance requires the exponents} \quad a \text{and} \quad i\omega \text{to be equal. For real} \quad a \text{this only happens if} \quad a = \omega = 0. \]

27 If you took the imaginary part \( y = \Im z \) of the complex solution to \( z' - az = Re^{i(\omega t - \phi)} \), what equation would \( y(t) \) solve? Answer first with \( \phi = 0 \).

\[ \text{Solution} \quad \text{Assuming} \quad a \text{is real, the imaginary part of} \quad z' - az = Re^{i(\omega t - \phi)} \text{is the equation} \quad y' - ay = R \sin(\omega t - \phi). \text{With} \quad \phi = 0 \text{this is} \quad y' - ay = R \sin \omega t. \]

Problems 28-31 solve first order circuit equations: not RLC but RL and RC.
Chapter 1. First Order Equations

28 Solve \( L \frac{dI}{dt} + RI(t) = V \cos \omega t \) for the current \( I(t) = I_n + I_p \) in the RL loop.

Solution Divide the equation by \( L \) to produce \( \frac{dI}{dt} - aI = X \cos \omega t \) with \( a = -\frac{R}{L} \) and \( X = \frac{V}{L} \). In this standard form, equation (3) gives the real solution:

\[
I = M \cos \omega t + N \sin \omega t \quad \text{with} \quad M = -\frac{aX}{\omega^2 + a^2} \quad \text{and} \quad N = \frac{\omega X}{\omega^2 + a^2}.
\]

29 With \( L = 0 \) and \( \omega = 0 \), that equation is Ohm’s Law \( V = IR \) for direct current. The complex impedance \( Z = R + i\omega L \) replaces \( R \) when \( L \neq 0 \) and \( I(t) = I e^{i\omega t} \).

\[
L \frac{dI}{dt} + RI(t) = (i\omega L + R)Ie^{i\omega t} = V e^{i\omega t} \quad \text{gives} \quad Z I = V.
\]

What is the magnitude \( |Z| = |R + i\omega L| \)? What is the phase angle in \( Z = |Z|e^{i\theta} \)? Is the current \( |I| \) larger or smaller because of \( L \)?

Solution \( |Z| = \sqrt{R^2 + \omega^2 L^2} \) and \( \tan \theta = \frac{\omega L}{R} \).

Since \( |Z| \) increases with \( L \), the current \( |I| \) must decrease.

30 Solve \( R \frac{dq}{dt} + \frac{1}{C} q(t) = V \cos \omega t \) for the charge \( q(t) = q_n + q_p \) in the RC loop.

Solution Dividing by \( R \) produces \( \frac{dq}{dt} - aq = X \cos \omega t \) with \( a = -\frac{1}{RC} \) and \( X = \frac{V}{R} \).

As in Problem 28, equation (3) gives \( M \) and \( N \) from \( \omega \) and this \( a \).

31 Why is the complex impedance now \( Z = R + \frac{1}{i\omega C} \)? Find its magnitude \( |Z| \).

Note that mathematics prefers \( i = \sqrt{-1} \), we are not conceding yet to \( j = \sqrt{-1} \)!

Solution The physical RC equation for the current \( I = \frac{dq}{dt} \) is \( RI + \frac{1}{C} \int I dt = V \cos \omega t = \text{Re} (V e^{i\omega t}) \).

The solution \( I \) has the same frequency factor \( X e^{i\omega t} \), and the integral has the factor \( e^{i\omega t} / i\omega \). Substitute into the equation and match coefficients of \( e^{i\omega t} \):

\[
RX + \frac{1}{i\omega C} X = V \quad \text{is} \quad Z X = V \quad \text{with impedance} \quad Z = R + \frac{1}{i\omega C}.
\]

Problem Set 1.6, page 50

1 Solve the equation \( dy/dt = y + 1 \) up to time \( t \), starting from \( y(0) = 4 \).

Solution We use the formula \( y(t) = y(0)e^{at} + \frac{s}{a}(e^{at} - 1) \) with \( a = 1 \) and \( s = 1 \) and \( y(0) = 4 \):

\[
y(t) = 4e^t + e^t - 1 = 5e^t - 1.
\]

2 You have $1000 to invest at rate \( a = 1 = 100\% \). Compare after one year the result of depositing \( y(0) = 1000 \) immediately with no source \( (s = 0) \), or choosing \( y(0) = 0 \) and \( s = 1000/year \) to deposit continually during the year. In both cases \( dy/dt = y + q \).

Solution We substitute the values for the different scenarios into the solution formula:

\[
y(t) = 1000e^t \quad \text{at one year}
\]

\[
y(t) = 1000e^t - 1000 = 1000(e - 1) \quad \text{at one year}
\]

You get more for depositing immediately rather than during the year.
3 If \(\frac{dy}{dt} = y - 1\), when does your original deposit \(y(0) = \frac{1}{2}\) drop to zero?

**Solution** Again we use the equation \(y(t) = y(0)e^{at} + \frac{s}{a}(e^{at} - 1)\) with \(a = 1\) and \(s = -1\). We set \(y(t) = 0\) and find the time \(t\):

\[
y(t) = y(0)e^t - e^t + 1 = e^t(y(0) - 1) + 1 = 0
\]

\[
e^t = \frac{1}{1 - y(0)} = 2 \quad \text{and} \quad t = \ln 2.
\]

Notice! If \(y(0) > 1\), the balance never drops to zero. Interest exceeds spending.

4 Solve \(\frac{dy}{dt} = y + t^2\) from \(y(0) = 1\) with increasing source term \(t^2\).

**Solution** Solution formula (12) with \(a = 1\) and \(y(0) = 1\) gives

\[
y(t) = e^t + \int_0^t e^{t-s}s^2ds = e^t - t(t + 2) + 2e^t - 2 = 3e^t - t(t + 2) - 2
\]

Check: \(\frac{dy}{dt} = 3e^t + 2t - 2\) equals \(y + t^2\).

5 Solve \(\frac{dy}{dt} = y + e^t\) (resonance \(a = e^t\)) from \(y(0) = 1\) with exponential source \(e^t\).

**Solution** The solution formula with \(a = 1\) and source \(e^t\) (resonance!) gives:

\[
y(t) = e^t + \int_0^t e^{t-s}e^sds = e^t + \int_0^t e^t ds = e^t(1 + t)
\]

Check by the product rule: \(\frac{dy}{dt} = e^t(1 + t) + e^t = y + e^t\).

6 Solve \(\frac{dy}{dt} = y - t^2\) from an initial deposit \(y(0) = 1\). The spending \(q(t) = -t^2\) is growing. When (if ever) does \(y(t)\) drop to zero?

**Solution**

\[
y(t) = e^t - \int_0^t e^{t-s}s^2ds = e^t + t(t + 2) - 2e^t + 2 = -e^t + t(t + 2).
\]

This definitely drops to zero (I regret there is no nice formula for that time \(t\)).

Check: \(\frac{dy}{dt} = -e^t + 2t + 2 = y - t^2\).

7 Solve \(\frac{dy}{dt} = y - e^t\) from an initial deposit \(y(0) = 1\). This spending term \(-e^t\) grows at the same \(e^t\) rate as the initial deposit (resonance). When (if ever) does \(y(t)\) drop to zero?

**Solution** \(y(t) = e^t - \int_0^t e^{t-s}e^s ds = e^t - \int_0^t e^t ds = e^t(1 - t)\) (this is zero at \(t = 1\))

Check by the product rule: \(\frac{dy}{dt} = e^t(1 - t) - e^t = y - e^t\).
8 Solve \( \frac{dy}{dt} = y - e^{2t} \) from \( y(0) = 1 \). At what time \( T \) is \( y(T) = 0 \)?

**Solution**

\[
\begin{align*}
  y(t) &= e^t - \int_0^t e^{t-s} e^{2s} \, ds = e^t - \int_0^t e^{t+s} \, ds = e^t + e^t(1 - e^t) = 2e^t - e^{2t}
\end{align*}
\]

This solution is zero when \( 2e^t = e^{2t} \) and \( 2 = e^t \) and \( t = \ln 2 \).

Check that \( y = 2e^t - e^{2t} \) solves the equation:

\[
\frac{dy}{dt} = 2e^t - 2e^{2t} = y - e^{2t}.
\]

9 Which solution \((y \text{ or } Y)\) is eventually larger if \( y(0) = 0 \) and \( Y(0) = 0 \)?

\[
\begin{align*}
  \frac{dy}{dt} &= y + 2t & \text{or} & & \frac{dY}{dt} &= 2Y + t.
\end{align*}
\]

**Solution**

\[
\begin{align*}
  y(t) &= \int_0^t e^{t-s} \cdot 2s \, ds \\
  Y(t) &= \int_0^t e^{2t-2s} \cdot s \, ds
\end{align*}
\]

\[
\begin{align*}
  y(t) &= 2(-t + e^t - 1) \\
  Y(t) &= \frac{e^{2t} - 1}{2}
\end{align*}
\]

In the long run \( Y(t) \) is larger than \( y(t) \), since the exponent \( 2t \) is larger than \( t \).

10 Compare the linear equation \( y' = y \) to the separable equation \( y' = y^2 \) starting from \( y(0) = 1 \). Which solution \( y(t) \) must grow faster? It grows so fast that it blows up to \( y(T) = \infty \) at what time \( T \)?

**Solution**

\[
\begin{align*}
  \frac{dy}{dt} &= y \\
  \frac{dy}{y} &= dt \\
  \int_{y(0)}^{y(t)} \frac{du}{u} &= \int_0^t dt \\
  \ln(y(t)) - \ln(y(0)) &= t
\end{align*}
\]

\[
\begin{align*}
  y(t) &= e^t \\
  \frac{y(t)}{y(0)} &= e^t
\end{align*}
\]

The second solution grows much faster, and reaches a vertical asymptote at \( T = 1 \).

11 \( Y' = 2Y \) has a larger growth factor (because \( a = 2 \)) than \( y' = y + q(t) \). What source \( q(t) \) would be needed to keep \( y(t) = Y(t) \) for all time?

**Solution**

\[
\begin{align*}
  \frac{dY}{dt} &= 2Y + 1 \quad \text{with for example} \quad Y(0) = y(0) = 0
\end{align*}
\]

\[
\begin{align*}
  Y(t) &= \int_0^t e^{2t-2s} \, ds = \frac{e^{2t} - 1}{2}
\end{align*}
\]
1.6. Models of Growth and Decay

Put this solution into \( \frac{dy}{dt} = y + q(t) \):

\[
e^{2t} = \frac{e^{2t} - 1}{2} + q(t)
\]

\[
e^{2t} = \frac{e^{2t} + 1}{2} = q(t)
\]

12 Starting from \( y(0) = Y(0) = 1 \), does \( y(t) \) or \( Y(t) \) eventually become larger?

\[
dy \over dt = 2y + e^t
\]

\[
y(t) = e^{2t} + \int_0^t e^{2t-2s}e^s\,ds = e^{2t} + e^{2t} - e^t = 2e^{2t} - e^t
\]

Solving the second equation:

\[
dY \over dt = Y + e^{2t}
\]

\[
Y(t) = e^t + \int_0^t e^{t-s}e^{2s}\,ds = e^t + e^{2t} - e^t = e^{2t} \quad \text{is always smaller than} \quad y(t).
\]

Questions 13-18 are about the growth factor \( G(s, t) \) from time \( s \) to time \( t \).

13 What is the factor \( G(s, s) \) in zero time? Find \( G(s, \infty) \) if \( a = -1 \) and if \( a = 1 \).

Solution The solution doesn’t change in zero time so \( G(s, s) = 1 \). (Note that the integral of \( a(t) \) from \( s \) to \( t = s \) is zero. Then \( G(s, s) = e^0 = 1 \). We are talking about change in the null solution, with \( y' = a(t)y \). A source term with a delta function does produce instant change.)

If \( a = -1 \), the solution drops to zero at \( t = \infty \). So \( G(s, \infty) = 0 \).

If \( a = 1 \), the solution grows infinitely large as \( t \to \infty \). So \( G(s, \infty) = \infty \).

14 Explain the important statement after equation (13): The growth factor \( G(s, t) \) is the solution to \( y' = a(t)y + \delta(t-s) \). The source term \( \delta(t-s) \) deposits \$1 at time \( s \).

Solution When the source term \( \delta(t-s) \) deposits \$1 at time \( s \), that deposit will grow or decay to \( y(t) = G(s, t) \) at time \( t > s \). This is consistent with the main solution formula (13).

15 Now explain this meaning of \( G(s, t) \) when \( t \) is less than \( s \). We go backwards in time. For \( t < s \), \( G(s, t) \) is the value at time \( t \) that will grow to equal \( 1 \) at time \( s \).

When \( t = 0 \), \( G(s, 0) \) is the “present value” of a promise to pay \$1 at time \( s \). If the interest rate is \( a = 0.1 = 10\% \) per year, what is the present value \( G(s, 0) \) of a million dollar inheritance promised in \( s = 10 \) years?

Solution In fact \( G(t, s) = 1/G(s, t) \). In the simplest case \( y' = y \) of exponential growth, \( G(s, t) \) is the growth factor \( e^{t-s} \) from \( s \) to \( t \). Then \( G(t, s) \) is \( e^{s-t} = 1/e^{t-s} \).

That number \( G(t, s) \) would be the “present value” at the earlier time \( t \) of a promise to pay \$1 at the later time \( s \). You wouldn’t need to deposit the full \$1 because your deposit will grow by the factor \( G(s, t) \). All you need to have at the earlier time is \( 1/G(s, t) \), which then grows to 1.
16 (a) What is the growth factor $G(s, t)$ for the equation $y' = (\sin t)y + Q \sin t$?
(b) What is the null solution $y_n = G(0, t)$ to $y' = (\sin t)y$ when $y(0) = 1$?
(c) What is the particular solution $y_p = \int_0^t G(s, t) Q \sin s \, ds$?

Solution (a) Growth factor: $G(s, t) = \exp \left( \int_s^t \sin T \, dT \right) = \exp(\cos s - \cos t)$.
(b) Null solution: $y_n = G(0, t) y(0) = e^{1 - \cos t}$.
(c) Particular solution: $y_p = \int_0^t e^{\cos s - \cos t} Q \sin s \, ds$

$= Q e^{-\cos t} [-e^{\cos s}]_0^t = Q (e^{1 - \cos t} - 1)$. Check $y_p(0) = Q(e^0 - 1) = 0$.

17 (a) What is the growth factor $G(s, t)$ for the equation $y' = y/(t + 1) + 10$?
(b) What is the null solution $y_n = G(0, t)$ to $y' = y/(t + 1)$ with $y(0) = 1$?
(c) What is the particular solution $y_p = 10 \int_0^t G(s, t) \, ds$?

Solution (a) $G(s, t) = \exp \left[ \int_s^t \frac{dT}{T + 1} \right] = \exp [\ln(t + 1) - \ln(s + 1)] = \frac{t + 1}{s + 1}$.
Null solution $y_n = G(0, t) y(0) = \exp [\ln(t + 1)] = t + 1$ since $\ln(0 + 1) = 0$.
Particular solution $y_p = 10 \int_0^t \exp [\ln(t + 1) - \ln(s + 1)] \, ds = 10(t + 1) \int_0^t \frac{ds}{s + 1} = 10(t + 1) \ln(t + 1)$.

18 Why is $G(t, s) = 1/G(s, t)$? Why is $G(s, t) = G(s, S)G(S, t)$?

Solution Multiplying $G(s, t) G(t, s)$ gives the growth factor $G(s, s)$ from going up to time $t$ and back to time $s$. This factor is $G(s, s) = 1$. So $G(t, s) = 1/G(s, t)$.
Multiplying $G(s, S) G(S, t)$ gives the growth factor $G(s, t)$ from going up from $s$ to $S$ and continuing from $S$ to $t$. In the example $y' = y$, this is $e^{S-s} e^{t-S} = e^{t-s} = G(s, t)$.

Problems 19–22 are about the “units” or “dimensions” in differential equations.

19 (recommended) If $dy/dt = ay + qe^{i\omega t}$, with $t$ in seconds and $y$ in meters, what are the units for $a$ and $q$ and $\omega$?

Solution $a$ is in “inverse seconds”—for example $a = .01$ per second.
$q$ is in meters.
$\omega$ is in “inverse seconds” or 1/seconds—for example $\omega = 2\pi$ radians per second.
20 The logistic equation \( \frac{dy}{dt} = ay - by^2 \) often measures the time \( t \) in years (and \( y \) counts people). What are the units of \( a \) and \( b \) ?

Solution \( a \) is in “inverse years”—for example \( a = 1 \) percent per year.

\( b \) is in “inverse people-years” as in \( b = 1 \) percent per person per year.

21 Newton’s Law is \( m \frac{d^2y}{dt^2} + ky = F \). If the mass \( m \) is in grams, \( y \) is in meters, and \( t \) is in seconds, what are the units of the stiffness \( k \) and the force \( F \) ?

Solution \( ky \) has the same units as \( m \frac{d^2y}{dt^2} \) so \( k \) is in grams per (second)^2. \( F \) is in gram-meters per (second)^2—the units of force.

22 Why is our favorite example \( y' = y + 1 \) very unsatisfactory dimensionally? Solve it anyway starting from \( y(0) = -1 \) and from \( y(0) = 0 \).

The three terms in \( y' = y + 1 \) seem to have different units. The rate \( a = 1 \) is hidden (with its units of 1/time). Also hidden are the units of the source term 1.

Solution \( y(t) = y(0)e^t + \frac{1}{a}(e^t - 1) \). This is \( e^t - 1 \) if \( y(0) = 0 \). The solution stays at steady state if \( y(0) = -1 \).

23 The difference equation \( Y_{n+1} = cY_n + Q_n \) produces \( Y_1 = cY_0 + Q_0 \). Show that the next step produces \( Y_2 = c^2Y_0 + cQ_0 + Q_1 \). After \( N \) steps, the solution formula for \( Y_N \) is like the solution formula for \( y' = ay + q(t) \). Exponentials of \( a \) change to powers of \( c \), the null solution \( e^{at}y(0) \) becomes \( c^N Y_0 \). The particular solution

\[
Y_N = c^{N-1}Q_0 + \cdots + Q_{N-1}
\]

is like \( y(t) = \int_{0}^{t} e^{a(t-s)}q(s)ds \).

Solution \( Y_2 = cY_1 + Q_1 = c(cY_0 + Q_0) + Q_1 = c^2Y_0 + cQ_0 + Q_1 \).

The particular solution \( cQ_0 + Q_1 \) agrees with the general formula when \( N = 2 \). The null solution \( c^2Y_0 \) is Step 2 in \( Y_0, cY_0, c^2Y_0, c^3Y_0, \ldots \) like \( e^{at}y(0) \).

24 Suppose a fungus doubles in size every day, and it weighs a pound after 10 days. If another fungus was twice as large at the start, would it weigh a pound in 5 days?

Solution This is an ancient puzzle and the answer is 9 days. Starting twice as large cuts off 1 day.

### Problem Set 1.7, page 61

1 If \( y(0) = a/2b \), the halfway point on the S-curve is at \( t = 0 \). Show that \( d = b \) and \( y(t) = \frac{a}{d} e^{-at} + b \). Sketch the classic S-curve — graph of \( 1(e^{-at} + 1) \) from \( y_{-\infty} = 0 \) to \( y_{\infty} = \frac{a}{b} \). Mark the inflection point.

Solution \( d = \frac{a}{y(0)} - b \) and \( y(0) = \frac{a}{2b} \) lead to \( d = \frac{a}{2b} - b = 2b - b = b \)

Therefore \( y(t) = \frac{a}{d} e^{-at} + b = \frac{a}{b} e^{-at} + b = \frac{a}{b} e^{-at} + 1 \)
2 If the carrying capacity of the Earth is \( K = \frac{a}{b} = 14 \) billion people, what will be the population at the inflection point? What is \( \frac{dy}{dt} \) at that point? The actual population was 7.14 billion on January 1, 2014.

**Solution** The inflection point comes where \( y = \frac{a}{2b} = 7 \) million. The slope \( \frac{dy}{dt} \) is

\[
\frac{dy}{dt} = ay - by^2 = a \left( \frac{a}{2b} \right) - b \left( \frac{a}{2b} \right)^2 = \frac{a^2}{4b}. \text{ This is } b \left( \frac{a}{2b} \right)^2 = 49b.
\]

3 Equation (18) must give the same formula for the solution \( y(t) \) as equation (16). If the right side of (18) is called \( R \), we can solve that equation for \( y \):

\[
y = R \left( 1 - \frac{b}{a}y \right) \rightarrow \left( 1 + \frac{b}{a} \frac{R}{y} \right) y = R \rightarrow y = \frac{R}{1 + \frac{R}{a} y}.
\]

Simplify that answer by algebra to recover equation (16) for \( y(t) \).

**Solution** This problem asks us to complete the partial fractions method which integrates \( \frac{dy}{(y - \frac{b}{a}y^2)} = \int dt \). The result in equation (18) can be solved for \( y(t) \). The right side of (18) is called \( R \):

\[
R = e^{at} \frac{y(0)}{1 - \frac{b}{a}y(0)} = e^{at} \frac{y(0)}{a - by(0)} = e^{at} \frac{a}{d}.
\]

Then the algebra in the problem statement gives

\[
y = \frac{R}{1 + \frac{R}{a} y} = \frac{e^{at} \frac{a}{d}}{1 + e^{at} \frac{a}{d}} = \text{ multiply by } \frac{de^{-at}}{de^{-at}} = \frac{a}{de^{-at} + b}.
\]

4 Change the logistic equation to \( y' = y + y^2 \). Now the nonlinear term is positive, and cooperation of \( y \) with \( y \) promotes growth. Use \( z = 1/y \) to find and solve a linear equation for \( z \), starting from \( z(0) = y(0) = 1 \). Show that \( y(T) = \infty \) when \( e^{-T} = 1/2 \). Cooperation looks bad, the population will explode at \( t = T \).

**Solution** Put \( y = 1/z \) and the chain rule \( \frac{dy}{dt} = -\frac{1}{z^2} \frac{dz}{dt} \) into the cooperation equation \( y' = y + y^2 \):

\[
-\frac{1}{z^2} \frac{dz}{dt} = \frac{1}{z} + \frac{1}{z^2} \text{ gives } \frac{dz}{dt} = -z - 1.
\]

The solution starting from \( z(0) = 1 \) is \( z(t) = 2e^{-t} - 1 \). This is zero when \( 2e^{-T} = 1 \) or \( e^T = 2 \) or \( T = \ln 2 \).

At that time \( z(T) = 0 \) means \( y(T) = 1/z(T) \) is infinite: blow-up at time \( T = \ln 2 \).

5 The US population grew from 313,873,685 in 2012 to 316,128,839 in 2014. If it were following a logistic \( S \)-curve, what equations would give you \( a, b, d \) in the formula (4)? Is the logistic equation reasonable and how to account for immigration?

**Solution** We need a third data point to find all three numbers \( a, b, d \). See Problem (23). There seems to be no simple formula for those numbers. Certainly the logistic equation is too simple for serious science. Immigration would give a negative value for \( h \) in the harvesting equation \( y' = ay - by^2 - h \).
6 The Bernoulli equation \( y' = ay - by^n \) has competition term \( by^n \). Introduce \( z = y^{1-n} \) which matches the logistic case when \( n = 2 \). Follow equation (4) to show that \( z' = (n - 1)(-az + b) \). Write \( z(t) \) as in (5)-(6). Then you have \( y(t) \).

**Solution** We make the suggested transformation:
\[
\begin{align*}
  z &= y^{1-n} \\
  z' &= (1-n)y^{-n}y' \\
  \frac{dz}{dt} &= (1-n)y^{-n}(ay - by^n) = (1-n)(ay^{1-n} - b) \\
  \frac{dz}{dt} &= (1-n)(az - b) \\
  z(t) &= e^{(1-n)at} - \frac{b}{a}e^{(1-n)at} - 1 = \frac{de^{(1-n)at} + b}{a} \\
  d &= az(0) - b = -\frac{a}{y(0)} - b \\
  y(t) &= e^{(1-n)at} + b
\end{align*}
\]

**Problems 7–13** develop better pictures of the logistic and harvesting equations.

7 \( y' = y - y^2 \) is solved by \( y(t) = 1/(de^{-t} + 1) \). This is an \( S \)-curve when \( y(0) = 1/2 \) and \( d = 1 \). But show that \( y(t) \) is very different if \( y(0) > 1 \) or if \( y(0) < 0 \).

If \( y(0) = 2 \) then \( d = \frac{1}{2} - 1 = -\frac{1}{2} \). Show that \( y(t) \to 1 \) from above.

If \( y(0) = -1 \) then \( d = \frac{1}{2} - 1 = -2 \). At what time \( T \) is \( y(T) = -\infty \) ?

**Solution** First, \( y(0) = 2 \) is above the steady-state value \( y_\infty = a/b = 1/1 \). Then \( d = -\frac{1}{2} \) and \( y(t) = 1/(1 - 1/2e^{-t}) \) is larger than 1 and approaches \( y(\infty) = 1/1 \) from above as \( e^{-t} \) goes to zero.

Second, \( y(0) = -1 \) is below the \( S \)-curve growing from \( y(-\infty) = 0 \) to \( y(\infty) = 1 \). The value \( d = -2 \) gives \( y(t) = 1/(-2e^{-t} + 1) \). When \( e^{-t} \) equals \( 1/2 \) this is \( y(t) = 1/0 \) and the solution blows up. That blowup time is \( t = \ln 2 \).

8 (recommended) Show those 3 solutions to \( y' = y - y^2 \) in one graph! They start from \( y(0) = 1/2 \) and 2 and \(-1 \). The \( S \)-curve climbs from \( 1/2 \) to 1. Above that, \( y(t) \) descends from 2 to 1. Below the \( S \)-curve, \( y(t) \) drops from \(-1 \) to \(-\infty \).

Can you see 3 regions in the picture? **Dropoff curves above** \( y = 1 \) and **\( S \)-curves sandwiched between** 0 and 1 and **dropoff curves below** \( y = 0 \).

**Solution** The three curves are drawn in Figure 3.3 on page 157. The upper curves and middle curves approach \( y_\infty = a/b \). The lowest curves reach \( y = -\infty \) in finite time: blow-up.

9 Graph \( f(y) = y - y^2 \) to see the unstable steady state \( Y = 0 \) and the stable \( Y = 1 \). Then graph \( f(y) = y - y^2 - 2/9 \) with harvesting \( h = 2/9 \). What are the steady states \( Y_1 \) and \( Y_2 \)? The 3 regions in Problem 8 now have \( Z \)-curves above \( y = 2/3 \), \( S \)-curves sandwiched between 1/3 and 2/3, dropoff curves below \( y = 1/3 \).

**Solution** The steady states are the points where \( Y - Y^2 = 0 \) (logistic) and \( Y - Y^2 - \frac{2}{9} = 0 \) (harvesting). That second equation factors into \( (Y - \frac{1}{3})(Y - \frac{2}{3}) \) to show the steady states \( \frac{1}{3} \) and \( \frac{2}{3} \).
10 What equation produces an $S$-curve climbing to $y_\infty = K$ from $y_{-\infty} = L$?

Solution We can choose $y' = ay - by^2 - h$ with steady states $K$ and $L$. Then $aK - bK^2 - h = 0$ and $aL - bL^2 - h = 0$. If we divide by $h$, these two linear equations give

\[
\frac{a}{h} = \frac{K + L}{KL} = \frac{1}{K} + \frac{1}{L} \quad \text{and} \quad \frac{b}{h} = \frac{1}{KL}
\]

Check: $\frac{a}{h}K - \frac{b}{h}K^2 - 1 = \frac{K}{L} = \frac{K}{L} = 0$ and $\frac{a}{h}L - \frac{b}{h}L^2 - 1 = \frac{L}{K} - \frac{L}{K} = 0$

11 $y' = y - y^2 - \frac{1}{t} = -(y - \frac{1}{2})^2$ shows critical harvesting with a double steady state at $y = Y = \frac{1}{2}$. The layer of $S$-curves shrinks to that single line. Sketch a drop in curve that starts above $y(0) = \frac{1}{2}$ and a drop off curve that starts below $y(0) = \frac{1}{2}$.

Solution The solution to $y' = -(y - \frac{1}{2})^2$ comes from integrating $-dy/(y - \frac{1}{2})^2 = dt$ to get $1/(y - \frac{1}{2}) = t + C$. Then $y(t) = \frac{1}{2} + \frac{1}{t+C}$. If $y(0) > \frac{1}{2}$ then $C > 0$ and this curve approaches $y(\infty) = \frac{1}{2}$; it is a hyperbola coming down toward that horizontal line. If $y(0) < \frac{1}{2}$ then $C$ is negative and the above solution $y = \frac{1}{2} + \frac{1}{t+C}$ blows up (or blows down! since $y$ is negative) at the positive time $t = -C$. This is a drop off curve below the horizontal line $y = \frac{1}{2}$. (If $y(0) = \frac{1}{2}$ the equation is $dy/dt = 0$ and the solution stays at that steady state.)

12 Solve the equation $y' = -(y - \frac{1}{2})^2$ by substituting $v = y - \frac{1}{2}$ and solving $v' = -v^2$.

Solution This approach uses the solutions we know to $dv/dt = -v^2$. Those solutions are $v(t) = \frac{1}{1+2t}$. Then $v = y - \frac{1}{2}$ gives the same $y = \frac{1}{2} + \frac{1}{1+2t}$ as in Problem 11.

13 With overharvesting, every curve $y(t)$ drops to $-\infty$. There are no steady states. Solve $Y - Y^2 - h = 0$ (quadratic formula) to find only complex roots if $4h > 1$.

The solutions for $h = \frac{5}{4}$ are $y(t) = \frac{1}{2} - \tan(t + C)$. Sketch that drop off if $C = 0$. Animal populations don’t normally collapse like this from overharvesting.

Solution Overharvesting is $y' = y - y^2 - h$ with $h$ larger than $\frac{1}{4}$ (Problems 11 and 12 had $h = \frac{1}{4}$ and critical harvesting). The fixed points come from $Y - Y^2 - h = 0$. The quadratic formula gives $Y = \frac{1}{2}(1 \pm \sqrt{1 - 4h})$. These roots are complex for $h > \frac{1}{4}$:

No fixed points.

For $h = \frac{5}{4}$ the equation is $y' = y - y^2 - \frac{5}{4} = -(y - \frac{1}{2})^2 - 1$. Then $v = y - \frac{1}{2}$ has $v' = -v^2 - 1$. Integrating $dv/(1 + v^2) = -dt$ gives $\tan^{-1}v = -t - C$ or $v = -\tan(t + C)$. $y = v + \frac{1}{2} = \frac{1}{2} - \tan(t + C)$. The graph of $-\tan t$ starts at zero and drops to $-\infty$ at $t = \pi/2$.

14 With two partial fractions, this is my preferred way to find $A = \frac{1}{r - s}$, $B = \frac{1}{s - r}$

\[
\begin{align*}
\frac{1}{(y - r)(y - s)} &= \frac{1}{(y - r)(r - s)} + \frac{1}{(y - s)(s - r)}
\end{align*}
\]

Check that equation: The common denominator on the right is $(y - r)(y - s)(r - s)$. The numerator should cancel the $r - s$ when you combine the two fractions.
1.7. The Logistic Equation

Separate $\frac{1}{y^2 - 1}$ and $\frac{1}{y^2 - y}$ into two fractions $\frac{A}{y - r} + \frac{B}{y - s}$.

Note When $y$ approaches $r$, the left side of PF2 has a blowup factor $1/(y - r)$. The other factor $1/(y - s)$ correctly approaches $A = 1/(r - s)$. So the right side of PF2 needs the same blowup at $y = r$. The first term $A/(y - r)$ fits the bill.

Solution

$$\frac{1}{y^2 - 1} = \frac{1}{(y - 1)(y + 1)} = \frac{A}{y - 1} + \frac{B}{y + 1} = \frac{1/2}{y - 1} - \frac{1/2}{y + 1}$$

The constants are $A = \frac{1}{r - s} = \frac{1}{1 - (-1)} = -\frac{1}{2} = -B$

$$\frac{1}{y^2 - y} = \frac{1}{(y - 1)y} = \frac{A}{y - 1} + \frac{B}{y} = \frac{1}{y - 1} - \frac{1}{y}, \quad A = \frac{1}{r - s} = \frac{1}{1 - 0} = -B$$

15 The threshold equation is the logistic equation backward in time:

$$-\frac{dy}{dt} = ay - by^2$$

is the same as $\frac{dy}{dt} = -ay + by^2$.

Now $Y = 0$ is the stable steady state. $Y = a/b$ is the unstable steady state (why?). If $y(0)$ is below the threshold $a/b$ then $y(t) \to 0$ and the species will die out.

Graph $y(t)$ with $y(0) < a/b$ (reverse S-curve). Then graph $y(t)$ with $y(0) > a/b$.

Solution The steady states of $\frac{dy}{dt} = -ay + by^2$ come from $-aY + bY^2 = 0$ so again $Y = 0$ or $Y = a/b$. The stability is controlled by the sign of $\frac{df}{dy}$ at $y = Y$:

$$f = -ay + by^2$$
tells how $y$ grows $\quad \frac{df}{dy} = -a + 2by$ tells how $\Delta y$ grows

$Y = 0$ has $\frac{df}{dy} = -a$ (STABLE) $\quad Y = \frac{a}{b}$ has $\frac{df}{dy} = -a + 2b \left( \frac{a}{b} \right) = a$ (UNSTABLE)

The S-curves go downward from $Y = a/b$ toward the line $Y = 0$ (never touch).

16 (Cubic nonlinearity) The equation $y'' = y(1 - y)(2 - y)$ has three steady states: $Y = 0, 1, 2$. By computing the derivative $\frac{df}{dy}$ at $y = 0, 1, 2$, decide whether each of these states is stable or unstable.

Draw the stability line for this equation, to show $y(t)$ leaving the unstable Y’s.

Sketch a graph that shows $y(t)$ starting from $y(0) = \frac{1}{2}$ and $\frac{1}{2}$ and $\frac{3}{2}$.

Solution $y' = f(y) = y(1 - y)(2 - y) = 2y - 3y^2 + y^3$ has slope $\frac{df}{dy} = 2 - 6y + 3y^2$.

$Y = 0$ has $\frac{df}{dy} = 2$ (unstable) $\quad$ S-curves go up from $Y = 0$ toward $Y = 1$

$Y = 1$ has $\frac{df}{dy} = -1$ (stable) $\quad$ S-curves from $Y = 2$ go down toward $Y = 1$

$Y = 2$ has $\frac{df}{dy} = 2$ (unstable) $\quad$ $y$
17 (a) Find the steady states of the **Gompertz equation** \( dy/dt = y(1 - \ln y) \).

**Solution**

(a) \( Y(1 - \ln Y) = 0 \) at steady states \( Y = 0 \) and \( Y = e \).

(b) Show that \( z = \ln y \) satisfies the linear equation \( dz/dt = 1 - z \).

**Solution**

\[ z = \ln y \text{ has } \frac{dz}{dt} = \frac{1}{y} \frac{dy}{dt} = y(1 - \ln y)/y = 1 - \ln y = 1 - z. \]

(c) The solution \( z(t) = 1 + e^{-t}(z(0) - 1) \) gives what formula for \( y(t) \) from \( y(0) \)?

**Solution**

(c) \( z' = 1/z \) gives that \( z(t) \). Then set \( y(t) = 1/z(t) \):

\[ y(t) = \left[1 + e^{-t}(z(0) - 1)\right]^{-1} = \left[1 + e^{-t}\left(\frac{1}{y(0)} - 1\right)\right]^{-1}. \]

18 Decide stability or instability for the steady states of

(a) \( dy/dt = 2(1 - y) (1 - e^y) \)  
(b) \( dy/dt = (1 - y^2)(4 - y^2) \)

**Solution**

(a) \( f(y) = 2(1 - y)(1 - e^y) = 0 \) at \( Y = 1 \) and \( Y = 0 \)

\[ \frac{df}{dy} = -2e^y(1 - y) - 2(1 - e^y) \]

At \( Y = 1 \) \[ \frac{df}{dy} = -2(1 - e) > 0 \text{ (UNSTABLE)} \] At \( Y = 0 \) \[ \frac{df}{dy} = -2 \text{ (STABLE)} \]

(b) \( f(y) = (1 - y^2)(4 - y^2) = 0 \) at \( Y = 1, -1, 2, -2 \)

\[ \frac{df}{dy} = -10y + 4y^3 \]

\( Y = 1 \) gives \( \frac{df}{dy} = -6 \text{ (STABLE)} \) \( Y = -1 \) gives \( \frac{df}{dy} = 6 \text{ (UNSTABLE)} \)

\( Y = 2 \) gives \( \frac{df}{dy} = 12 \text{ (UNSTABLE)} \) \( Y = -2 \) gives \( \frac{df}{dy} = -12 \text{ (STABLE)} \)

19 Stefan’s Law of Radiation is \( dy/dt = K(M^4 - y^4) \). It is unusual to see fourth powers. Find all real steady states and their stability. Starting from \( y(0) = M/2 \), sketch a graph of \( y(t) \).

**Solution**

\( f(y) = K(M^4 - y^4) \) equals 0 at \( Y = M \) and \( Y = -M \) (also \( Y = \pm iM \)).

\[ \frac{df}{dy} = -4KY^3 = -4KM^4(Y = M \text{ is STABLE}) \quad \frac{df}{dy} = 4KM^4(Y = -M \text{ is UNSTABLE}) \]

The graph starting at \( y(0) = M/2 \) must go upwards to approach \( y(\infty) = M \).

20 \( dy/dt = ay - y^3 \) has how many steady states \( Y \) for \( a < 0 \) and then \( a > 0 \)? Graph those values \( Y(a) \) to see a **pitchfork bifurcation**—new steady states suddenly appear as \( a \) passes zero. The graph of \( Y(a) \) looks like a pitchfork.

**Solution**

\( f(y) = ay - y^3 = y(a - y^2) \) has 3 steady states \( Y = 0, \sqrt{a}, -\sqrt{a} \).

\[ \frac{df}{dy} = a - 3y^2 \text{ equals } a \text{ at } Y = 0, \quad \frac{df}{dy} = -2a \text{ at } Y = \sqrt{a} \text{ and } Y = -\sqrt{a}. \]

Then \( Y = 0 \) is UNSTABLE and \( Y = \pm \sqrt{a} \) are STABLE.

21 (Recommended) The equation \( dy/dt = \sin y \) has **infinitely many steady states**. What are they and which ones are stable? Draw the stability line to show whether \( y(t) \) increases or decreases when \( y(0) \) is between two of the steady states.

**Solution**

\( f(Y) = \sin Y \) is zero at every steady state \( Y = n\pi \ (0, \pi, -\pi, 2\pi, -2\pi, \ldots) \)

\[ \frac{df}{dy} = \cos Y = 1 \text{ (UNSTABLE for } Y = 0, 2\pi, -2\pi, 4\pi, \ldots) \]

\[ \frac{df}{dy} = \cos Y = -1 \text{ (STABLE for } Y = \pi, -\pi, 3\pi, -3\pi, \ldots) \]

**Stability line**

```
-2\pi  -\pi  0  \pi  2\pi
-\sqrt{a}  \sqrt{a}  \sqrt{a}  -\sqrt{a}
```
22 Change Problem 21 to \( \frac{dy}{dt} = (\sin y)^2 \). The steady states are the same, but now the derivative of \( f(y) = (\sin y)^2 \) is zero at all those states (because \( \sin y \) is zero). What will the solution actually do if \( y(0) \) is between two steady states?

Solution \( f(y) = (\sin y)^2 \) has \( \frac{df}{dy} = 2 \sin y \cos y = \sin 2y. \)

Now \( \frac{df}{dy} = 0 \) at all the steady states \( Y = n\pi. \)

Since \( \frac{df}{dy} = (\sin y)^2 \) is always positive, the solution \( y(t) \) will always increase toward the next larger steady state.

We have an infinite stack of \( S \)-curves.

23 (Research project) Find actual data on the US population in the years 1950, 1980, and 2010. What values of \( a, b, d \) in the solution formula (7) will fit these values? Is the formula accurate at 2000, and what population does it predict for 2020 and 2100?

You could reset \( t = 0 \) to the year 1950 and rescale time so that \( t = 3 \) is 1980.

Solution Resetting time gives \( T = c(t - 1950) \). Rescaling gives \( c(1980 - 1950) = 3 \) so \( c = \frac{1}{10} \). Then \( a, b, d \) depend on your data.

The graphs from \( t = 1950 \) to 1980 will show \( T = \frac{1}{10}(t - 1950) \) from \( T = 0 \) to 3.

24 If \( \frac{dy}{dt} = f(y) \), what is the limit \( y(\infty) \) starting from each point \( y(0) \)?

Solution

\[
\frac{dy}{dt} = \begin{cases} y & \text{for } y \leq 1 \\ 2 - y & \text{for } y \geq 1 \end{cases}
\]

Slope \( \frac{df}{dy} = 1 \) at \( Y = 0 \) (UNSTABLE). Slope \( \frac{df}{dy} = -1 \) at \( Y = 2 \) (STABLE), \( y(\infty) = 2. \)

25 (a) Draw a function \( f(y) \) so that \( y(t) \) approaches \( y(\infty) = 3 \) from every \( y(0). \)

Solution The right side \( f(y) \) must be zero only at \( Y = 3 \) which is STABLE.

Example: \( \frac{dy}{dt} = f(y) = 3 - y \) has solutions \( y = 3 + Ce^{-t}. \)

(b) Draw \( f(y) \) so that \( y(\infty) = 4 \) if \( y(0) > 0 \) and \( y(\infty) = -2 \) if \( y(0) < 0. \)

Solution This requires \( Y = 4, -2 \) to be stable and \( Y = 0 \) to be unstable.

Example: \( \frac{dy}{dt} = f(y) = -y(y - 4)(y + 2) \) Notice \( \frac{df}{dy} = 8 \) at \( Y = 0. \)

26 Which exponents \( n \) in \( \frac{dy}{dt} = y^n \) produce blowup \( y(T) = \infty \) in a finite time? You could separate the equation into \( \frac{dy}{y^n} = dt \) and integrate from \( y(0) = 1. \)
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Chapter 1. First Order Equations

Solution \[ \int \frac{dy}{y^n} = \int dt \text{ gives } \frac{y^{1-n}}{1-n} = t + C. \] The right side is zero at a finite time \( t = -C \). Then \( y \) blows up at that time if \( n > 1 \).

If \( n = 1 \) the integrals give \( \ln y = t + C \) and \( y = e^{t+C} \); NO BLOWUP in finite time.

27 Find the steady states of \( \frac{dy}{dt} = y^2 - y^4 \) and decide whether they are stable, unstable, or one-sided stable. Draw a stability line to show the final value \( y(\infty) \) from each initial value \( y(0) \).

Solution \( f(y) = y^2 - y^4 = 0 \) at \( Y = 0, 1, -1 \)

\[ \frac{df}{dy} = 2y - 4y^3 = -2 \text{ at } Y = 1 \text{ (STABLE)} \]

\[ \frac{df}{dy} = 2y - 4y^3 = 2 \text{ at } Y = -1 \text{ (UNSTABLE)} \]

Since \( Y = -1 \) is unstable, \( y(t) \) must go toward \( Y = 0 \) if \( -1 < y(0) < 0 \).

Since \( Y = 1 \) is stable, \( y(t) \) must go toward \( Y = 1 \) if \( 0 < y(0) < 1 \).

28 For an autonomous equation \( y' = f(y) \), why is it impossible for \( y(t) \) to be increasing at one time \( t_1 \) and decreasing at another time \( t_2 \)?

Solution Reason: The stability line shows a movement of \( y \) in one direction, away from one (unstable) steady state \( Y \) and toward another (stable) steady state. “One direction” means that \( y(t) \) is steadily increasing or steadily decreasing.

Problem Set 1.8, page 69

1 Finally we can solve the example \( \frac{dy}{dt} = y^2 \) in Section 1.1 of this book.

Start from \( y(0) = 1 \). Then \( \int_1^y \frac{dy}{y^2} = \int_0^t dt \). Notice the limits on \( y \) and \( t \). Find \( y(t) \).

Solution With those limits, integration gives \( -\frac{1}{y} + 1 = t \). Then \( \frac{1}{y} = 1 - t \) and \( y(t) = \frac{1}{1-t} \).

2 Start the same equation \( \frac{dy}{dt} = y^2 \) from any value \( y(0) \). At what time \( t \) does the solution blow up? For which starting values \( y(0) \) does it never blow up?

Solution \( -\frac{1}{y} + \frac{1}{y(0)} = t \) gives \( \frac{1}{y} = \frac{1}{y(0)} - t \) and \( y = \frac{y(0)}{1 - ty(0)} \).

If \( y(0) \) is negative, then \( 1 - ty(0) \) never touches zero for \( t > 0 \): No blowup.

3 Solve \( \frac{dy}{dt} = a(t)y \) as a separable equation starting from \( y(0) = 1 \), by choosing \( f(y) = 1/y \). This equation gave the growth factor \( G(0,t) \) in Section 1.6.

Solution \( \int_{y(0)}^y \frac{dy}{y} = \int_0^t a(t) dt \) gives \( \ln y(t) - \ln y(0) = \int_0^t a(t) dt \)
1.8. Separable Equations and Exact Equations

\[ y(t) = y(0) \exp \left( \int_0^t a(t) \, dt \right) = G(0, t) \, y(0) \]

4 Solve these separable equations starting from \( y(0) = 0 \):

(a) \( \frac{dy}{dt} = ty \quad \text{ (b) } \frac{dy}{dt} = t^m y^n \)

Solution (a) \( \int \frac{dy}{y} = \int \frac{t \, dt}{y(0)} \) and \( \ln y - \ln y(0) = t^2 / 2 \) : Then \( y(t) = y(0) \exp(t^2 / 2) \).

(b) \( \frac{dy}{dt} = t^m y^n \) has \( \int \frac{dy}{y^n} = \int t^m \, dt \) and \( \frac{y^{1-n}}{1-n} = \frac{t^{m+1}}{m+1} \). Then \( y = \left( \frac{1-n}{m+1} \right)^{1/(1-n)} \) for \( n \neq 1 \).

5 Solve \( \frac{dy}{dt} = a(t) y^2 = \frac{a(t)}{1/y^2} \) as a separable equation starting from \( y(0) = 1 \).

Solution \( \frac{dy}{dt} = a(t) y^2 \)

\[ \int \frac{du}{u^2} = \int a(x) \, dx \quad (u \text{ and } x \text{ are just integration variables}) \]

\[ -\frac{1}{y} + 1 = \int_0^t a(x) \, dx \quad \text{gives} \quad y = \frac{1}{1 - \int_0^t a(x) \, dx} \]

6 The equation \( \frac{dy}{dt} = y + t \) is not separable or exact. But it is linear and \( y = \) _\_

Solution We solve the equation by taking advantage of its linearity:

Given \( a = 1 \), the growth factor is \( e^t \). The source term is \( t \). Therefore using equation (14) gives:

\[ y(t) = e^t y(0) + \int_0^t e^{t-s} \, ds = e^t y(0) - t + e^t - 1. \]

Check : \( dy/dt = e^t y(0) - 1 + e^t \) does equal \( y + t \).

7 The equation \( \frac{dy}{dt} = \frac{y}{t} \) has the solution \( y = At \) for every constant \( A \). Find this solution by separating \( f = 1/y \) from \( g = 1/t \). Then integrate \( dy/y = dt/t \). Where does the constant \( A \) come from ?

Solution We use separation of variables to find the constant \( A \)
Chapter 1. First Order Equations

\[ \frac{dy}{y} = \frac{dt}{t} \]

\[ \int_{y(1)}^{t} \frac{du}{u} = \int_{1}^{t} \frac{dx}{x} \]

\[ \ln(y) - \ln(y(1)) = \ln t \]

\[ \frac{y}{y(1)} = t \]

\[ y = y(1) t \]

Therefore we find that the constant \( A \) is equal to \( y(1) \), the initial value.

8 For which number \( A \) is \( \frac{dy}{dt} = \frac{ct - ay}{At + by} \) an exact equation? For this \( A \), solve the equation by finding a suitable function \( F(y, t) + C(t) \).

Solution \( f(y, t) = At + by \) and \( g(y, t) = ct - ay \)

The equation is exact if: \( \frac{\partial f}{\partial t} = -\frac{\partial g}{\partial y} \) and \( A = a \).

We follow the three solution steps for exact equations.

1 Integrate \( f \) with respect to \( y \):

\[ \int f(y, t) \, dy = \int (At + by) \, dy = Aty + \frac{1}{2}by^2 = F(y, t) \]

2 Choose \( C(t) \) so that \( \frac{\partial}{\partial t}(F(y, t) + C(t)) = -g(y, t) \)

\[ \frac{\partial}{\partial t}(Aty + \frac{1}{2}by^2 + C(t)) = Ay + C'(t) = -ct + ay \]

\[ C'(t) = -ct \text{ and } C(t) = -\frac{1}{2}ct^2 \]

3 We therefore have that:

\[ \frac{dy}{dt} = \frac{g(y, t)}{f(y, t)} \]

is solved by \( F(y, t) + C(t) = \text{ constant} \)

\[ Aty + \frac{1}{2}by^2 - \frac{1}{2}ct^2 = \text{ constant} \]

9 Find a function \( y(t) \) different from \( y = t \) that has \( \frac{dy}{dt} = \frac{y^2}{t^2} \).

Solution Using separation of variables:

\[ \frac{dy}{dt} = \frac{y^2}{t^2} \]

\[ \frac{dy}{y^2} = \frac{dt}{t^2} \]

\[ \int_{y(t_0)}^{y} \frac{du}{u^2} = \int_{t_0}^{t} \frac{dx}{x^2} \]

\[ -\frac{1}{y(t)} + \frac{1}{y(t_0)} = -\frac{1}{t} + \frac{1}{t_0} \]

\( t_0 = 1 \) and \( y(t_0) = 2 \) give \(-\frac{1}{y(t)} + \frac{1}{2} = -\frac{1}{t} + 1 \) and \( y(t) = \left( \frac{1}{t} - \frac{1}{2} \right)^{-1} = \frac{2t}{2-t} \)
These equations are separable after factoring the right hand sides:

\[
\begin{align*}
\frac{dy}{dt} &= e^{y+t} \quad \text{and} \quad \frac{dy}{dt} = yt + y + t + 1.
\end{align*}
\]

**Solution**

(a) \( \frac{dy}{dt} = e^{y} e^{t} \) and

\[
\int_{y_0}^{y} e^{-y} dy = \int_{t_0}^{t} e^{t} dt
\]

\[
-e^{-y} + e^{-y_0} = e^{t} - e^{t_0}
\]

\[
e^{-y} = e^{-y_0} - e^{t} + e^{t_0}
\]

\[
y = -\ln(e^{-y_0} - e^{t} + e^{t_0})
\]

(b) \( \frac{dy}{dt} = (y + 1)(t + 1) \)

\[
\int_{y_0}^{y} \frac{dy}{y + 1} = \int_{t_0}^{t} (t + 1) dt
\]

\[
\ln(y + 1) - \ln(y_0 + 1) = \frac{1}{2}(t^2 - t_0^2) + (t - t_0) = G
\]

\[
y + 1 = (y_0 + 1) e^G
\]

These equations are linear and separable:

\[
\frac{dy}{dt} = (y + 4) \cos t \quad \text{and} \quad \frac{dy}{dt} = ye^t.
\]

**Solution**

(a) \( \int_{y_0}^{y} \frac{dy}{y + 4} = \int_{t_0}^{t} \cos t dt \)

\[
\ln(y + 4) - \ln(y_0 + 4) = \sin t - \sin t_0
\]

\[
y + 4 = (y_0 + 4) \exp(\sin t - \sin t_0)
\]

(b) \( \int_{y_0}^{y} \frac{dy}{y} = \int_{t_0}^{t} e^t dt \)

\[
\ln y - \ln y_0 = e^t - e^{t_0}
\]

\[
y = y_0 \exp(e^t - e^{t_0})
\]

Solve these three separable equations starting from \( y(0) = 1 \):

(a) \( \frac{dy}{dt} = -4ty \) has \( \int_{1}^{y} \frac{dy}{y} = \int_{0}^{t} -4t dt \)

\[
\ln y = -2t^2 \quad \text{and} \quad y = \exp(-2t^2)
\]

(b) \( \frac{dy}{dt} = ty^3 \) has \( \int_{1}^{y} \frac{dy}{y^3} = \int_{0}^{t} t dt \) and \( -\frac{1}{2y^2} + \frac{1}{2y_0^2} = \frac{1}{2} t^2 \)
Chapter 1. First Order Equations

Test the exactness condition \( \partial g / \partial y \).

13 Test the exactness condition \( \partial g / \partial y = -\partial f / \partial t \) and solve Problems 13-14.

(c) \((1 + t) \frac{dy}{dt} = 4y\) has

\[
\begin{align*}
\frac{1}{y^2} & = \frac{1}{y_0^2} - t^2 \\
y & = \left(\frac{1}{y_0} - t^2\right)^{-1/2} = y_0 \left(1 - t^2 y_0^2\right)^{-1/2}
\end{align*}
\]

Check \((1 + t) \frac{dy}{dt} = 4(1 + t)(1 + t)^3 = 4y\)

14 Test the exactness condition \( \partial g / \partial y = -\partial f / \partial t \).

Solution (a) \( g = -3t^2 - 2y^2 \) has \( \partial g / \partial y = -4y \)

\[
f = 4ty + by^2 \quad \text{has} \quad -\partial f / \partial y = -4y : \text{EXACT}
\]

Step 1: \( \int f \, dy = \int (4ty + 6y^2) \, dy = 2ty^2 + 2y^3 + C(t) \)

Step 2: \( \frac{\partial}{\partial t} (2ty^2 + 2y^3 + C(t)) = 2y^2 + C'(t) \).

This equals \( -g \) when \( C'(t) = 3t^2 \) and \( C(t) = t^3 \).

Solution (b) \( g = -1 - ye^{ty} \) has \( \partial g / \partial y = -yte^{ty} - e^{ty} \)

\[
f = 2y + te^{ty} \text{ has } -\partial f / \partial t = -yte^{ty} - e^{ty} : \text{EXACT}
\]

Step 1: \( \int f \, dy = \int (2y + te^{ty}) \, dy = y^2 + e^{ty} + C(t) = F(y, t) \)

Step 2: \( \frac{\partial}{\partial t} (y^2 + e^{ty} + C(t)) = ye^{ty} + C'(t) = -g \) where \( C'(t) = 1 \)

Step 3: \( C'(t) = 1 \) gives \( C(t) = t \) and the solution is \( F(y, t) + C(t) = -yte^{ty} - e^{ty} + t = \text{constant} \)
1.8. Separable Equations and Exact Equations

Step 2: \( \frac{\partial}{\partial t} (2ty^2 + 2y^3 + C(t)) = 2y^2 + C'(t) \) when \( C' = 3t^2 \) and \( C = t^3 \)

Step 3: Solution \( 2ty^2 + 2y^3 + t^3 = \text{constant} \)

15 Show that \( \frac{dy}{dt} = -\frac{y^2}{2ty} \) is exact but the same equation \( \frac{dy}{dt} = -\frac{y}{2t} \) is not exact. Solve both equations. (This problem suggests that many equations become exact when multiplied by an integrating factor.)

Solution \( g = -y^2 \) and \( f = 2ty \) have \( \frac{\partial g}{\partial y} = -2y = -\frac{\partial f}{\partial t} : \text{EXACT} \)

\( g = -y \) and \( f = 2t \) have \( \frac{\partial g}{\partial y} \) NOT EQUAL TO \( -\frac{\partial f}{\partial t} \)

Solve the second form which is SEPARABLE

\[ \int \frac{dy}{y} = \int \frac{-dt}{2t} \text{ gives } \ln y = -\frac{1}{2} \ln t + C \]

Then \( y = e^{C_1 t^{-1/2}} \) is the same as \( y = ct^{-1/2} \).

The same solution must come from Steps 1, 2, 3 using the exact form.

16 Exactness is really the condition to solve two equations with the same function \( H(t, y) \):

\[ \frac{\partial H}{\partial y} = f(t, y) \text{ and } \frac{\partial H}{\partial t} = -g(t, y) \]

can be solved if \( \frac{\partial f}{\partial t} = -\frac{\partial g}{\partial y} \).

Take the \( t \) derivative of \( \frac{\partial H}{\partial y} \) and the \( y \) derivative of \( \frac{\partial H}{\partial t} \) to show that exactness is necessary. It is also sufficient to guarantee that a solution \( H \) will exist.

Solution The point is to see the underlying idea of exactness.

If \( \frac{\partial H}{\partial y} = f(t, y) \) then \( \frac{\partial^2 H}{\partial t \partial y} = \frac{\partial f}{\partial t} \)

If \( \frac{\partial H}{\partial t} = -g(t, y) \) then \( \frac{\partial^2 H}{\partial y \partial t} = -\frac{\partial g}{\partial y} \)

The cross derivatives of \( H \) are always equal. IF a function \( H \) solves both equations then \( \frac{\partial f}{\partial t} \) must equal \( -\frac{\partial g}{\partial y} \). So behind every exact equation is a function \( H \) : exactness is a necessary and also sufficient to find \( H \) with \( \frac{\partial H}{\partial y} = f \) and \( \frac{\partial H}{\partial t} = -g \).

17 The linear equation \( \frac{dy}{dt} = aty + q \) is not exact or separable. Multiply by the integrating factor \( e^{-\int at \, dt} \) and solve the equation starting from \( y(0) \).

Solution This problem just recalls the idea of an integrating factor:

For \( \frac{dy}{dt} = aty + q \) the factor is \( P = \exp \left( -\int at \, dt \right) = \exp \left( -\frac{1}{2}at^2 \right) \).

Then \( P \left( \frac{dy}{dt} - aty \right) \) agrees with \( (Py)' = P \frac{dy}{dt} + \frac{dP}{dt} y \)

So the original equation multiplied by \( P \) is \( \frac{d}{dt}(Py) = Pq \).

Integrate both sides \( P(t)y(t) - P(0)y(0) = \int_0^t P(t)q \, dt \). Divide by \( P(t) \) to find \( y(t) \).
Chapter 1. First Order Equations

Second order equations \( F(t, y, y', y'') = 0 \) involve the second derivative \( y'' \). This reduces to a first order equation for \( y' \) (not \( y \)) in two important cases:

| I. | When \( y \) is missing in \( F \), set \( y' = v \) and \( y'' = v' \). Then \( F(t, v, v') = 0 \). |
| II. | When \( t \) is missing in \( F \), set \( y'' = dv/dt = dv/dy dy/dt \). Then \( F(y, v, v dv/dy) = 0 \). |

See the website for reduction of order when one solution \( y(t) \) is known.

18 (\( y \) is missing) Solve these differential equations for \( v = y' \) with \( v(0) = 1 \). Then solve for \( y \) with \( y(0) = 0 \).

**Solution** (a) \( y'' + y' = 0 \). Set \( y' = v \). Then \( v' + v = 0 \) gives \( v(t) = Ce^{-t} \).

Now solve \( y' = v = Ce^{-t} \) to find \( y = Ce^{-t} + D \).

**Solution** (b) \( 2ty'' - y' = 0 \). Set \( y' = v \). Then \( 2tv' - v = 0 \) is solved by

\[
\int \frac{dv}{v} = \int \frac{dt}{2t} \quad \text{and} \quad \ln v = \ln \sqrt{t} + C \quad \text{and} \quad v = c\sqrt{t}.
\]

Now solve \( y' = v = c\sqrt{t} \) to find \( y = \frac{c}{6}t^{3/2} + D \).

19 Both \( y \) and \( t \) are missing in \( y'' = (y')^2 \). Set \( v = y' \) and go two ways:

I. Solve \( \frac{dv}{dt} = v^2 \) to find \( v = \frac{1}{1-t} \) as in Section 1.1.

Then solve \( \frac{dy}{dt} = v = \frac{1}{1-t} \) to find \( y = -(1-t)^{-2} + \frac{1}{2} \) with \( y(0) = 0 \).

II. Solve \( \frac{dv}{dy} = v^2 \) or \( \frac{dv}{dy} = v \) to find \( v = \exp y \).

Then \( \frac{dy}{dt} = v(y) = \exp y \) gives \( \int e^{-y} dy = \int dt \) satisfying \( v(0) = 1, y(0) = 0 \)

and \( -e^{-y} = t - 1 \); not the same solution as part I (?)

20 An autonomous equation \( y' = f(y) \) has no terms that contain \( t \) (\( t \) is missing). Explain why every autonomous equation is separable. A non-autonomous equation could be separable or not. For a linear equation we usually say LTI (linear time-invariant) when it is autonomous: coefficients are constant, not varying with \( t \).

**Solution** Every autonomous equation separates into

\[
\int \frac{dy}{f(y)} = \int dt.
\]

Linear equations can be \( \frac{dy}{dt} = a(t) y \) : Non-autonomous

LTI equations are \( \frac{dy}{dt} = ay \) (linear and also \( a \) is time-invariant \( \Rightarrow \) autonomous).

21 \( my'' + ky = 0 \) is a highly important LTI equation. Two solutions are \( \cos \omega t \) and \( \sin \omega t \) when \( \omega^2 = k/m \). Solve differently by reducing to a first order equation for \( y' = dy/dt = v \) with \( y'' = v dv/dy \) as above:

\[
mv \frac{dv}{dy} + ky = 0 \quad \text{integrates to} \quad \frac{1}{2}mv^2 + \frac{1}{2}ky^2 = \text{constant } E.
\]
For a mass on a spring, kinetic energy \( \frac{1}{2}mv^2 \) plus potential energy \( \frac{1}{2}ky^2 \) is a constant energy \( E \). What is \( E \) when \( y = \cos \omega t \)? What integral solves the separable \( m(y')^2 = 2E - ky^2 \)? I would not solve the linear oscillation equation this way.

**Solution** With \( y' = v \) and \( y'' = \frac{dv}{dy} \), the equation \( my'' + ky = 0 \) becomes

\[
mv \frac{dv}{dy} + ky = 0.
\]

This is nonlinear but separable. Integrate \( mv \, dv = -ky \, dy \) to get

\[
\frac{1}{2}mv^2 + \frac{1}{2}ky^2 = \text{constant } E \quad \text{[Conservation of Energy].}
\]

If \( y = \cos(\omega t) \) then \( v = y' = -\omega \sin(\omega t) \) and \( E \) is \( \frac{1}{2}m \cos^2(\omega t) + \frac{1}{2}K\omega^2 \sin^2(\omega t) \).

The separable equation \( m(y')^2 = 2E - ky^2 \) could be solved by

\[
\left( \frac{m}{2E - Ky^2} \right)^{1/2} \, dy = dt.
\]

The integral could lead to \( \cos^{-1} y = \omega t \) and \( y = \cos \omega t \).

22. \( my'' + k \sin y = 0 \) is the nonlinear oscillation equation: not so simple. Reduce to a first order equation as in Problem 21:

\[
mv \frac{dv}{dy} + k \sin y = 0 \quad \text{integrates to } \frac{1}{2}mv^2 - k \cos y = \text{constant } E.
\]

With \( v = dy/dt \) what impossible integral is needed for this first order separable equation? Actually that integral gives the period of a nonlinear pendulum—this integral is extremely important and well studied even if impossible.

**Solution** Take square roots in \( \frac{1}{2}m \left( \frac{dy}{dt} \right)^2 = K \cos y + E \).

Then separate into

\[
\left( \frac{m/2}{K \cos y + E} \right)^{1/2} \, dy = dt.
\]

An unpleasant integral but important for nonlinear oscillation. Chapter 1 is ending with an example that shows the reality of nonlinear differential equations: Numerical solutions possible, elementary formulas are often impossible.