THE CORE IDEAS IN OUR TEACHING

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What will our students remember? One answer comes quickly but it is a counsel of despair: *nothing at all.* At the other extreme is an impossible hope that we all cherish: *everything we say.* Let me look for an intermediate answer, closer to reality, possibly by changing the question.

I have come to believe that each course has a central core. We may not see it ourselves, when we teach a new topic every day. For the calculus course, I won’t even venture an answer: at least not here. My examples will be differential equations and linear algebra, because writing a textbook forced me to uncover (painfully slowly!) the underlying structure of the course.

May I begin with linear algebra. The ideas of a vector space and a basis for that space are central. We have a serious job to help students understand these words. The building blocks are “linear combinations” and “linear independence”. We certainly need good examples, and good bases for them. I think it is here that the course becomes coherent—or it can scatter into unconnected examples of isolated ideas.

I will start with a matrix $A$. A more abstract person would start from a linear transformation. But we are aiming for a basis; we are choosing coordinates; they bring us to a matrix. There are four fundamental subspaces associated with that matrix:

1. Its nullspace $N(A)$ (the kernel) dimension $n - r$
2. Its column space $C(A)$ (the range) $r$
3. Its row space, which is $C(A^T)$ $r$
4. The nullspace $N(A^T)$ of the transpose $m - r$

These are the spaces that we want students to remember. I draw them as often as possible (two in $\mathbb{R}^n$ and two in $\mathbb{R}^m$). I count their basis vectors to find their dimension: the first big theorems in linear algebra. The rank $r$ determines all dimensions. I propose multiple choices of $A$—the beauty of this subject is in the wonderful variety of matrices. And I connect the four subspaces to factorizations of $A,$
which are really choices of bases that lie at the absolute center of pure and applied linear algebra.

The bases in $U$ and $Q$ and $S$ and $V$ become increasingly perfect.

\[
\begin{align*}
A &= LU & \text{Elimination gives an echelon basis for the row space} \\
A &= QR & \text{Gram-Schmidt gives an orthonormal basis for } C(A) \\
A &= SAS^{-1} & \text{Eigenvectors give a basis in which } A \text{ is diagonal} \\
A &= U\Sigma V^T & \text{Orthonormal bases in the columns of } U \text{ and } V.
\end{align*}
\]

We are constantly constructing bases for the fundamental subspaces. Elimination and Gram-Schmidt orthogonalization end after finitely many steps. Diagonalization by eigenvectors is deeper and better, but $A$ must be square and nondefective. The Singular Value Decomposition produces perfect bases $v_i$ and $u_i$ for all four subspaces – orthonormal and also diagonalizing for every matrix $A$:

\[
Av_i = \sigma_i u_i \quad (i \leq r) \quad Av_i = 0 \quad A^T u_i = 0 \quad (i > r)
\]

The success of the SVD comes from the spectral theorem for symmetric matrices: $A^T A$ has a full set of orthonormal eigenvectors $v_i$. Beautifully, the $u_i$ turn out to be orthonormal eigenvectors of $AA^T$. This can be a highlight for the last day of a linear algebra course.

For an earlier day, one idea is to ask students to “read” a few matrices:

\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
-1 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\]

The rotation is familiar, the projection is almost too easy. The difference matrix is also the incidence matrix for a simple graph (three nodes in a line). Incidence matrices of a larger graph are terrific examples – all four subspaces have a meaning.

May I turn from subspaces to the basic course on differential equations. Part of this course is a collection of methods to solve separable equations, exact equations, logistic equations $y' = ay - by^2$, and more. We go forward to systems of equations, and test nonlinear equations for stability. But the coherent part (the central problem) is to solve linear equations with constant coefficients. How can we present their solutions?

I believe we have to answer this question. It is the ODE equivalent of solving $Ax = 0$ and $Ax = b$ and $Ax = \lambda x$. It certainly rests on the most important functions in this course: exponentials $e^{st}$ and $e^{\lambda t}$. By
working with exponentials, we (almost) turn the differential equation into algebra.

Start with the simplest right hand sides \( f(t) = 0 \) and \( e^{st} \).

\[
Ay'' + By' + Cy = 0 \quad Ay'' + By' + Cy = e^{st}
\]

The key idea is to expect solutions \( y = Ge^{st} \):

\[
G(As^2 + Bs + C)e^{st} = 0 \quad G(As^2 + Bs + C)e^{st} = e^{st}.
\]

On the left, two values of \( s \) are allowed: the roots \( s_1 \) and \( s_2 \) of \( As^2 + Bs + C = 0 \). On the right, any \( s \) is allowed (and the possibilities \( s_1 = s_2 \) and \( s = s_1 = s_2 \) need special attention). Normally we have

\[
y_n = y_{\text{nullspace}} = c_1e^{s_1t} + c_2e^{s_2t}
\]

\[
y_p = y_{\text{particular}} = G(s)e^{st} = \frac{1}{As^2 + Bs + C}e^{st}.
\]

Those two parts of \( y(t) \) connect linear differential equations to linear algebra. The complete solution combines all \( y_n \) with one \( y_p \). Linearity is in control and the consequence is \( y = y_n + y_p \).

I apologize for asking you to read what you know so well. The simplicity of \( y = Ge^{st} \) has to be recognized and remembered. This is where calculus meets algebra. \( G \) is the prime example of an undetermined coefficient (determined by the equation). An elementary course could continue as far as \( f(t) = e^{i\omega t} \) and \( \cos \omega t \) and \( \sin \omega t \) and stop. The serious question is to solve the differential equation for all \( f(t) \).

I see two instructive ways to reach \( y(t) \). Both begin with special right hand sides, and combine the solutions. The combination has to be an integral and not just a finite sum: calculus is needed now. Here are the good options:

1. Combine exponentials \( e^{st} \) with weights \( F(s) \) to get \( f(t) \). By linearity, the solution \( y(t) \) will combine the exponentials \( F(s)G(s)e^{st} \).

2. Combine impulses \( \delta(t - s) \) with weights \( f(s) \) to get \( f(t) \). By linearity, the solution \( y(t) \) will combine the impulse responses \( f(s)g(t - s) \).

Where \( e^{st} \) is localized at frequency \( s \), the delta function \( \delta(t - s) \) is completely localized at time \( s \).
Method 1 uses the Laplace transform. The transform of \( f(t) \) gives the right weights \( F(s) \):

\[ F(s) = \text{transform of } f(t) \quad y(t) = \text{inverse transform of } F(s)G(s)e^{st}. \]

For each \( s \), that solution \( F(s)G(s)e^{st} \) is easy. The hard part is the inverse Laplace transform, to combine those solutions into \( y(t) \).

Realistically, we know a very limited number of transform pairs. Method 1 almost limits us to the same short list as before: \( f \) can combine \( e^{(a+i\omega)t}, \cos \omega t, \sin \omega t, t, \) and their products. This is a space of functions whose derivatives stay in the space. You can guess that I am advocating Method 2. It begins with an impulse \( \delta(t) \):

\[(1) \quad Ag'' + Bg' + Cg = \delta(t) \quad \text{with } g(0) = 0 \quad \text{and } g'(0) = 0.\]

Introducing that delta function is a good thing! We are finding the fundamental solution \( g(t) \)—the Green’s function, the growth factor, the impulse response. This is a high point in the course. And it is easy to do, because this same \( g(t) \) also solves the homogeneous equation:

\[(2) \quad Ag'' + Bg' + Cg = 0 \quad \text{with } g(0) = 0 \quad \text{and } g'(0) = 1/A.\]

The solution must have the form \( g(t) = c_1e^{s_1t} + c_2e^{s_2t}. \) The two initial conditions give \( c_1 \) and \( c_2 \) and a neat formula for \( g(t) \):

\[(3) \quad g(t) = \frac{e^{s_1t} - e^{s_2t}}{A(s_1 - s_2)} \left( \text{or } g(t) = \frac{te^{s_1t}}{A} \text{ when } s_1 = s_2 \right).\]

Then the original equation, with any right side \( f(t) \), is solved by

\[(4) \quad y_{\text{particular}}(t) = \int_0^t g(t-s) f(s) \, ds.\]

Discussion In coming quickly to the formula for \( y(t) \), I have left multiple loose ends. Let me go backwards more slowly, as we would certainly do in a classroom. Methods 1 and 2 are closely connected. The Laplace transform of \( \delta(t) \) is 1. Then equation (1) transforms to

\[(5) \quad (As^2 + Bs + C) \ G(s) = 1.\]

The transfer function \( G(s) = 1/(As^2 + Bs + C) \) is the Laplace transform of the impulse response \( g(t) \). These functions can be written in terms of \( A, B, C \) or \( s_1 \) or \( s_2 \). A lot of effort has gone into choosing good parameters! The damping ratio \( B/\sqrt{4AC} \) and the natural frequency \( \sqrt{C/A} \) are two of the best.
We must also explain why equations (1) and (2) have the same solution \( g(t) \). Mechanically, this comes from partial fractions:

\[
\frac{1}{As^2 + Bs + C} = \frac{1}{A(s - s_1)(s - s_2)} = \frac{1}{A(s_1 - s_2)} \left( \frac{1}{s - s_1} - \frac{1}{s - s_2} \right).
\]

The inverse Laplace transform confirms that \( e^{s_1 t} \) and \( e^{s_2 t} \) go into \( g(t) \).

Here is a truly “mechanical” explanation of (1) = (2). A bat hits a ball at \( t = 0 \). The velocity jumps instantly to \( g'(0) = 1/A \). This comes from integrating \( Ag'' +Bg' + Cg = \delta(t) \) from \( t = 0 \) to \( t = h \). The left side produces the jump in \( Ag' \) and the integral of \( \delta(t) \) is 1. The other terms disappear as \( h \to 0 \), leaving \( Ag'(0) = 1 \).

In working with \( \delta(t) \), some faith is needed. It is worth developing and it is not misplaced. A delta function is an extremely useful model. So is its integral the step function, which turns on a switch at \( t = 0 \). By linearity, the step response is the integral of \( g(t) \).

Finally, let me connect Method 1 directly to Method 2. In the first method, the Laplace transform of \( y(t) \) is \( F(s)G(s) \). In the second method, \( y(t) \) is the convolution of \( f(t) \) with \( g(t) \). The connection is the Convolution Rule: The transform of a convolution \( f(t) * g(t) \) is a multiplication \( F(s)G(s) \).

In the language of signal processing, any constant coefficient linear equation can be solved in the “s-domain” or the “t-domain.” The poles \( s_1, s_2 \) of the transfer function \( G(s) = \frac{1}{(As^2 + Bs + C)} \) control the behavior of \( y(t) \) : oscillation, decay, or instability. The whole course develops out of the quadratic formula for those roots \( s_1 \) and \( s_2 \).

Note The actual course would start with first order equations:

\[
y' - ay = 0 \quad y' - ay = e^{st}
\]

The null solutions are \( y_n = ce^{at} \). The particular solution is \( y_p = e^{st}/(s-a) \). The transfer function is \( G(s) = 1/(s-a) \). The fundamental solution (impulse response, growth factor, Green’s function) solves

\[
\begin{align*}
g' - ag &= \delta(t) \quad \text{with} \quad g(0) = 0 \\
g' - ag &= 0 \quad \text{with} \quad g(0) = 1
\end{align*}
\]

This function is simply \( g = e^{at} \). At this early point it doesn’t need all those names! We recognize it as \( 1/(\text{integrating factor}) \). Its Laplace transform is \( G(s) = 1/(s-a) \). For systems \( y' = Ay \) we have the matrix
exponential $g = e^{At}$. The solution $y_n + y_p$ for any right hand side $f(t)$ and initial condition $y(0)$ is

$$y(t) = y(0)e^{at} + \int_0^t e^{a(t-s)} f(s) \, ds.$$  

The input $f(s)$ at time $s$ grows in the remaining time $t-s$ by the factor $e^{a(t-s)}$. The solution $y(t)$ (the integral) combines all of these outputs $e^{a(t-s)} f(s)$.

That single paragraph translates into weeks of teaching, even without $\delta(t)$. Perhaps first order equations with constant coefficients might be the one topic that is understood and remembered? I don’t like to think so, because a teacher has to remain an optimist.

I plan to prepare video lectures going at a normal pace, and linked to [http://math.mit.edu/dela](http://math.mit.edu/dela). That website has much more about differential equations and linear algebra and a new textbook for those courses.