Rainbow Solutions to the Sidon Equation

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Abstract

We prove that for every 4-coloring of \( \{1, 2, \ldots, n\} \), with each color class having cardinality more than \( \frac{2n}{3} \), there exists a solution of the equation \( x + y = z + w \) with \( x, y, z \) and \( w \) belonging to different color classes. The lower bound on a color class cardinality is tight.

1 Introduction

Let \( \mathbb{N} \) denote the set of positive integers, and for \( i, j \in \mathbb{N}, i \leq j \), let \( [i, j] \) denote the set \( \{i, i+1, \ldots, j\} \) (with \( [n] \) abbreviating \( [1, n] \) as usual). One of the earliest results in Ramsey theory [8] is Schur’s theorem (1916) [17]: for every \( k \in \mathbb{N} \) and sufficiently large \( n \in \mathbb{N} \), every \( k \)-coloring of \( [n] \) contains a monochromatic solution of the equation \( x + y = z \). Another classical result in combinatorial number theory is due to van der Waerden (1927) [21]: for all \( m, k \in \mathbb{N} \), there is an integer \( n_0 = n_0(m, k) \), such that every \( k \)-coloring of \( [n] \), \( n \geq n_0 \), contains a monochromatic \( m \)-term arithmetic progression (abbreviated as \( AP(m) \) throughout). This statement was further generalized to sets of positive upper density in the celebrated work of Szemerédi [19] (see also [20]). Canonical versions of van der Waerden’s theorem were discovered by Erdős and others [7].

More than seven decades after Schur’s result, Alekseev and Savchev [1] considered what Bill Sands calls an un-Schur problem [9]. They proved that for every equinumerous 3-coloring of \( [3n] \) (i.e., a coloring in which different color classes have the same cardinality), the equation \( x + y = z \) has a solution with \( x, y \) and \( z \) belonging to different color classes. Such solutions will be called rainbow solutions. Esther Klein and George Szekeres asked whether the condition of equal cardinalities for three color classes can be weakened [18]. Indeed, Schönheim [16] proved that for every 3-coloring of \( [n] \), such that every color class has cardinality greater than \( n/4 \), the equation \( x + y = z \) has rainbow solutions. Moreover, he showed that \( n/4 \) is optimal.

Inspired by the un-Schur problem, Jungić et al. [10] sought a rainbow counterpart of van der Waerden’s theorem. Namely, given positive integers \( m \) and \( k \), what conditions on \( k \)-colorings of \( [n] \) guarantee the existence of an \( AP(m) \), all of whose elements have distinct colors? If every integer in \( [n] \) is colored by the largest power of three that divides it, then one immediately obtains a \( k \)-coloring of \( [n] \) with \( k \leq \lceil \log_3 n \rceil + 1 \) and without a rainbow \( AP(3) \). So, while Szemerédi’s theorem states that a large cardinality in only one color class ensures the existence of a monochromatic \( AP(m) \), one needs all color classes to be “large” to force a rainbow \( AP(m) \). In [10], it was proved that every 3-coloring of \( \mathbb{N} \) with the upper density of each color class greater than \( 1/6 \) yields a rainbow \( AP(3) \). Using some tools from additive number theory, they obtained similar (and stronger) results.

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for 3-colorings of $Z_n$ and $Z_p$, some of which were recently extended by Conlon [5]. The more difficult interval case was studied in [11], where it was shown that every equinumerous 3-coloring of $[3n]$ contains a rainbow $AP(3)$, that is, a rainbow solution to the equation $x + y = 2z$. Finally, Axenovich and Fon-Der-Flaass [2] cleverly combined the previous methods with some additional ideas to obtain the following theorem, conjectured in [10].

**Theorem 1** For every $n \geq 3$, every partition of $[n]$ into three color classes $\mathcal{R}$, $\mathcal{B}$, and $\mathcal{G}$ with $\min(|\mathcal{R}|, |\mathcal{B}|, |\mathcal{G}|) > r(n)$, where

$$r(n) := \begin{cases} \lfloor (n + 2)/6 \rfloor & \text{if } n \not\equiv 2 \pmod{6} \\ (n + 4)/6 & \text{if } n \equiv 2 \pmod{6} \end{cases},$$

contains a rainbow $AP(3)$.

The colorings

$$c(i) := \begin{cases} R & \text{if } i \equiv 1 \pmod{6} \\ B & \text{if } i \equiv 4 \pmod{6} \\ G & \text{otherwise} \end{cases} \quad \text{and} \quad \overline{c}(i) := \begin{cases} R & \text{if } i \leq \frac{n+1}{3} \text{ and } i \text{ is odd} \\ B & \text{if } i \geq \frac{2n+2}{3} \text{ and } i \text{ is even} \\ G & \text{otherwise} \end{cases}$$

show that Theorem 1 is the best possible for the cases $n \not\equiv 2 \pmod{6}$ and $n \equiv 2 \pmod{6}$, respectively. It is interesting to note that similar statements about the existence of rainbow $AP(k)$ in $k$-colorings of $[n]$, $k \geq 4$, do not hold [2, 6]. For example, the equinumerous 4-coloring $\lambda : [n] \rightarrow \{R, B, G, Y\}$

$$\lambda(i) := \begin{cases} R & \text{if } i \equiv 1 \pmod{4} \text{ and } i < 4m; \text{ or if } i \equiv 3 \pmod{4} \text{ and } i > 4m \\ B & \text{if } i \equiv 2 \pmod{4} \text{ and } i < 4m; \text{ or if } i \equiv 0 \pmod{4} \text{ and } i > 4m \\ G & \text{if } i \equiv 3 \pmod{4} \text{ and } i < 4m; \text{ or if } i \equiv 0 \pmod{4} \text{ and } i \leq 4m \\ Y & \text{if } i \equiv 1 \pmod{4} \text{ and } i > 4m; \text{ or if } i \equiv 2 \pmod{4} \text{ and } i \leq 4m \end{cases}$$

for every $i \in [n]$, $n = 8m$ ($m \in \mathbb{N}$), contains no rainbow $AP(4)$.

There are many directions and generalizations one can consider, such as searching for rainbow counterparts of other classical theorems in Ramsey theory [8, 12], increasing the number of colors or the length of a rainbow $AP$, or proving the existence of more than one rainbow $AP$. Some positive and negative results in these directions were obtained in [3, 4, 10].

In this paper, we study one such direction and consider the existence of rainbow solutions to other linear equations, imitating Rado’s theorem about the monochromatic analogue. Rado [15] called a rational matrix $A$ (or a system $Ax = 0$) $k$-partition regular if there exists an $n$ for which every $k$-coloring of $[n]$ has a monochromatic solution to the system of linear homogeneous equations $Ax = 0$. Furthermore, $A$ is called partition regular if it is $k$-partition regular for all $k$. Rado’s “columns condition” completely determines the matrices (or systems) which are partition regular. A special case of this theorem states that a single linear homogeneous equation $\sum_{i=1}^{m} a_i = 0$, $a_i \in \mathbb{Z}$ is partition regular if and only if some nonempty subset of the $a_i$’s sums to zero.

In particular, “the Sidon equation” $x + y = z + w$, a classical object in additive number theory [13, 14] is partition regular. In this note, we prove a rainbow analogue of this result.

**Theorem 2** For every $n \geq 4$, every partition of $[n]$ into four color classes $\mathcal{R}$, $\mathcal{B}$, $\mathcal{G}$, and $\mathcal{Y}$, with $\min(|\mathcal{R}|, |\mathcal{B}|, |\mathcal{G}|, |\mathcal{Y}|) > \frac{n+1}{6}$, contains a rainbow solution of $x + y = z + w$. Moreover, this result is tight.

One should contrast Theorem 2 with the aforementioned result of Conlon et al. [6], which states that there nevertheless exist equinumerous 4-colorings of $[n]$ with no rainbow $AP(4)$, i.e., with no rainbow solution of the system $x + y = z + w$, $x + w = 2z$. 

\[2\]
2 Proof of Theorem 2

We prove Theorem 2 for $n \geq 5$. Given partition of $[n]$ into four color classes $\mathcal{R}, \mathcal{B}, \mathcal{G},$ and $\mathcal{Y}$, with 
$$\min\{ |\mathcal{R}|, |\mathcal{B}|, |\mathcal{G}|, |\mathcal{Y}| \} > \frac{n+1}{6},$$
let $c : [n] \mapsto \{ R, G, B, Y \}$ be the corresponding coloring of $[n]$, i.e., 
$$\mathcal{R} = [n] \cap \{ i : c(i) = R \},$$
and similarly for $\mathcal{B}, \mathcal{G},$ and $\mathcal{Y}$. Suppose that there is no rainbow solution of the equation $x + y = z + w$.

We say that there is a string $s = c_1c_2\ldots c_m \in \{ R, G, B, Y \}^m$ at a position $i$ if $c(i) = c_1$, 
$c(i+1) = c_2$, \ldots, $c(i+m-1) = c_m$. We say that there is a string $s$ in the coloring $c$ if there is $s$ at some position $i$. We call a string bichromatic if it contains exactly two colors. A bichromatic string is complete if it cannot be extended (on either side) and still be bichromatic. Notice that since each color is used at least once, there are at least three complete bichromatic strings.

Since $c$ does not contain a rainbow solution of $x + y = z + w$, then there are no integers $a$, $b$, $d$, such that $a + b + d, b, a + d$ form a rainbow solution. In what follows, this observation will be denoted as the Q-property.

A particular color is called dominant if every bichromatic string contains that color. Clearly, if such a color exists, it will be unique. The first step in our proof is to establish the following claim.

**Lemma 1** $c$ contains a dominant color.

**Proof:** Consider the first two complete bichromatic strings, i.e., those with the least initial position. They share a common color. Without loss of generality, assume that the first bichromatic string contains colors $R$ and $B$, and the second bichromatic string contains colors $R$ and $Y$. In particular, $R$ and $B$, as well as $R$ and $Y$, occur next to each other. There exists at least one element of $[n]$ colored by $G$, and this element is contained in a bichromatic string. If the other color in the string is $B$ ($Y$), then $G$ and $B$ ($Y$) appear next to each other within this string. Since $R$ and $Y$ ($B$) are consecutive, we have a contradiction by the Q-property with $d = 1$. Therefore, every bichromatic string that contains $G$ also contains $R$.

Finally, suppose there is a bichromatic string with colors $B$ and $Y$. Then $B$ and $Y$ appear next to each other, and since $G$ and $R$ appear next to each other as well, we obtain a contradiction with the Q-property for $d = 1$.

We conclude that every bichromatic string contains $R$, and, therefore, $R$ is the dominant color. \(\square\)

Now, we can assume that $R$ is the dominant color in $c$. Let $d$ be the minimum distance between two differently colored non-red integers, that is 
$$d = \min\{ |x - y| : c(x) \neq c(y) \text{ and } x, y \notin \mathcal{R} \}.$$ 

Note that because $R$ is the dominant color, we have $d \geq 2$. Without loss of generality, assume that there exist two elements of $[n]$, distance $d$ apart, that are colored by $B$ and $Y$ respectively. By the Q-property, there do not exist two elements of $[n]$, distance $d$ apart, that are colored by $R$ and $G$ respectively. Next, we prove that every complete bichromatic string with colors $R$ and $G$ ($B$) has a special structure.

**Lemma 2** Let $X \in \{ G, B \}$. Every complete bichromatic string with colors $R$ and $X$ is $d$-periodic with exactly one element colored by $X$ within every substring of length $d$.

**Proof:** Consider a complete bichromatic string $s$ of length $m$ at a position $i$, with colors $R$ and $G$. The underlying interval $I = [i, i + m - 1]$ is the disjoint union of $I_k$, $0 \leq k \leq d - 1$, where
$I_k = \{ j \in [i, i + m - 1] \mid j \equiv k (\text{mod } d) \}$. By the Q-property, for every $0 \leq k \leq d - 1$, either all elements of $I_k$ are colored by $G$ or all elements of $I_k$ are colored by $R$.

Assume that $i \neq 1$. The case $i + m - 1 \neq n$ is symmetric and handled similarly. Let $g$ denote the smallest element of $I$ colored by $G$. If $g - d \geq i$ then $\{g, g - d\} \subseteq I_k$ for some $k \in \{0, 1, \ldots, d - 1\}$. So, $c(g - d) = G$, which contradicts our choice of $g$. Thus, $g - d < i$. Since $s$ is complete, $c(i - 1) \in \{B, Y\}$ and $g - (i - 1) \leq d$. Therefore, $g - d = i - 1$. Now, since $c(g - d) \in \{B, Y\}$, $c(g) = G$ and all the integers between $g - d$ and $g$ are colored by $R$, we conclude that all the elements of $I_k$, for $k \equiv g (\text{mod } d)$, are colored by $G$, while for all other values of $k \in \{1, \ldots, d\}$, all the elements of $I_k$ are colored by $R$.

Hence, from the above argument we see that every complete bichromatic string with colors $R$ and $G$ has the following structure: it is $d$-periodic with exactly one element colored by $G$ within every substring of length $d$. Moreover, since $c(g - d) \in \{B, Y\}$ and $g - d = i - 1$, it follows that we can assume, without loss of generality, that there exist two elements of $[n]$, distance $d$ apart, that are colored by $G$ and, say, $Y$, respectively. The previous argument then implies that every complete bichromatic string with colors $R$ and $B$ is $d$-periodic with exactly one element colored by $B$ within every substring of length $d$.

\[ \square \]

In particular, since $R$ is the dominant color, we obtain:

**Corollary 1** Strings $GG$ and $BB$ do not appear in $c$.

Now, the following claim is clear:

**Lemma 3** String $YY$ appears in $c$.

Proof: Suppose that $YY$ does not appear in $c$. Then, by Corollary 1, at least one in every pair of consecutive integers in $[n]$ would be colored by $R$. Therefore, $|R| \geq \lfloor \frac{n}{2} \rfloor \geq \frac{n - 1}{2}$, and $3\min(|Y|, |G|, |B|) \leq |Y| + |G| + |B| = n - |R| \leq \frac{n - 1}{2}$. So $\min(|Y|, |G|, |B|, |R|) \leq \frac{n + 1}{6}$, which contradicts our assumption.

\[ \square \]

**Lemma 4** $d = 2$.

Proof: Indeed, suppose that $d \geq 3$. By Lemma 2, we have $|R| \geq (d - 1)(|G| + |B|) - 1$. Then for $n \geq 5$, $n = |R| + |B| + |G| + |Y| \geq (d - 1)(\frac{n + 2}{6} + \frac{n + 2}{6} - 1) + \frac{n + 2}{2} > n$, which is a contradiction.

Lemma 2 and Lemma 4 imply the following claim:

**Corollary 2** Let $X \in \{G, B\}$. There exist two integers in $[n]$, with difference 2, that are colored by $X$ and $Y$, respectively. Furthermore, elements of every bichromatic string with colors $R$ and $X$ alternate in color.

**Lemma 5** Strings $BRG$ and $GRB$ do not appear in $c$.

Proof: Since (by Lemma 3) there is a string of at least two consecutive $Y$s and since $R$ is the dominant color in $c$, there are two integers in $[n]$, distance two apart, that are colored by $Y$ and $R$. The claim now follows from the Q-property. \[ \square \]
Lemma 6 At least one of the strings $GRG$ and $BRB$ appears in $c$.

Proof: Suppose that there is no $GRG$ nor $BRB$ in $c$. Let us consider four consecutive integers $i$, $i + 1$, $i + 2$, $i + 3$ in $\{1, \ldots, n\}$. If $c(i + 1) = G$, then $c(i) = c(i + 2) = R$, by the dominance of color $R$ and Corollary 1. Furthermore, $c(i + 3) \in \{R, Y\}$, by Lemma 5. If $c(i) = G$, then $c(i + 1) = R$, by the dominance of color $R$ and Corollary 1. Since $c(i) = G$ and $c(i + 1) = R$ belong to a dichromatic string with colors $R$ and $G$ (which alternates in color by Corollary 2), if we assume that $GRG$ does not appear in $c$, then $c(i + 2) = Y$, by Lemma 5. It follows that $c(i + 3) \in \{R, Y\}$.

Therefore, at most one integer in every string of length four can be colored by $B$ or $G$. We obtain $|G| + |B| \leq \left\lfloor \frac{n}{3} \right\rfloor$, and for $n \geq 5$, $\min(|R|, |G|, |B|, |Y|) \leq \min(|G|, |B|) \leq \frac{n+3}{8} \leq \frac{n+1}{6}$. This violates our condition on the minimum of color class cardinalities. □

By Lemma 6 we can assume that $GRG$ appears in $c$. By Lemma 3 there exists $p$, the smallest positive integer with the property that there is $i \in \{1, \ldots, n\}$ such that $c(i) \in \{G, B\}$ and at least one of the following is true:

$$(a) \quad c(i + p) = c(i + p + 1) = Y; \quad c(i + p - 1) = R; \quad c(i + j) \in \{R, Y\} \quad \text{for all} \quad 1 \leq j \leq p - 1 \quad \text{with} \quad R \in \{c(i + j), c(i + j + 1)\} \quad \text{for} \quad 1 \leq j \leq p - 2;$$

$$(b) \quad c(i - p) = c(i - p - 1) = Y; \quad c(i - p + 1) = R; \quad c(i - j) \in \{R, Y\} \quad \text{for all} \quad 1 \leq j \leq p - 1 \quad \text{with} \quad R \in \{c(i - j), c(i - j - 1)\} \quad \text{for} \quad 1 \leq j \leq p - 2.$$

Next, suppose that there is $i \in \{1, \ldots, n\}$ such that $c(i) = B$ and that, say, (a) is true. Let $m$ be such that $c(i + p + j) = Y$ for all $1 \leq j \leq m$ and, if $i + p + m + 1 \in \{1, \ldots, n\}$, $c(i + p + m + 1) = R$. Let $k \in \{1, \ldots, n\}$ be such that $c(k) = c(k + 2) = G$. We note that $k \notin \{i, i + p + m + 1\}$. Suppose $k > i + p + m + 1$. If $c(k - p) = R$, then $i, i + p, k - p, k$ contradict the $Q$-property. If $c(k - p) \in \{B, G, Y\}$, then $c(k - p + 1) = R$ and $i, i + p + 1, k - p + 1, k + 2$ contradict the $Q$-property.

Now, suppose that there is no $i \in \{1, \ldots, n\}$ such that $c(i) = B$ and that either (a) or (b) is true. Thus, there is $i \in \{1, \ldots, n\}$ such that $c(i) = G$ and that, say, (a) is true. Let $m$ be as above and let $\ell$ be an element from $B$ such that between $\ell$ and $[i, i + p + m + 1]$ there are no other elements from $B$. Suppose this time that $\ell < i$. If $c(\ell + p) = R$, then $\ell, \ell + p, i, i + p$ contradict the $Q$-property. If $c(\ell + p) \in \{G, Y\}$, then $c(\ell + p + 1) = R$ and $\ell, \ell + p + 1, i, i + p + 1$ contradict the $Q$-property (if $c(\ell + p) = Y$, then $c(\ell + p + 1) = R$, because of the minimality of $p$ and the assumption from the beginning of this paragraph).

In order to finish the proof of Theorem 2, we present a 4-coloring of $\{1, \ldots, n\}$ with the minimum size of a color class equal to $\left\lfloor \frac{n+1}{6} \right\rfloor$ and no rainbow solution of $x + y = z + w$:

$$c(i) := \begin{cases} B & \text{if } i \equiv 1 \pmod{6} \\ G & \text{if } i \equiv 3 \pmod{6} \\ Y & \text{if } i \equiv 5 \pmod{6} \\ R & \text{otherwise} \end{cases}$$

3 Concluding remarks

It is curious to note that the minimal “density” for the color classes is $\frac{1}{6}$ in Theorem 2, as well as in Theorem 1. It is also interesting to note that a dominant color exists when one studies the existence of rainbow solutions to equations $x + y = 2z$ or $x + y = z$ in the 3-colorings of $\{1, \ldots, n\}$ [2, 10, 11].
what other systems of equations does a rainbow-free coloring, under certain cardinality constraints, must have a dominant color?

The question of rainbow partition regularity is an interesting one. It would be exciting to provide a complete rainbow analogue of Rado’s theorem (which classified the partition regular matrices [15]). Theorem 2 is a small step in this direction.

We say a vector is rainbow if every entry of the vector is colored differently. A matrix $A$ with rational entries is called rainbow partition $k$-regular if for all $n$ and every equinumerous $k$-coloring of $[kn]$ there exists a rainbow vector $x$ such that $Ax = 0$. We say that $A$ is rainbow regular if there exists $k_1$ such that $A$ is rainbow partition $k$-regular for all $k \geq k_1$. For example, Theorem 2 shows that the following matrix is rainbow partition 4-regular:

$$A = \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}.$$

We let the rainbow number of $A$, denoted by $r(A)$, be the least $k$ for which $A$ is rainbow partition $k$-regular. It is not difficult to see that every $1 \times n$ matrix $A$ with nonzero entries is rainbow partition regular if and only if not all the entries in $A$ are of the same sign. It would be interesting to study the rainbow number $r(A)$. Furthermore, we somewhat boldly conjecture the following characterization of rainbow regularity.

**Conjecture 1** Matrix $A$ with integer entries is rainbow regular if and only if the rows of $A$ are linearly independent and there exists a vector $u$ with positive integer entries such that $Au = 0$.

Jungić et al. [10] prove that for every $k \geq 3$, $\lfloor \frac{k^2}{4} \rfloor < r(A) \leq \frac{k(k-1)^2}{2}$, where $A$ is the following $(k - 1) \times (k + 1)$ matrix:

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 & -2 & 1 \end{pmatrix}.$$

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**References**


