Problem 1. [2 points]
Prove that there is no graph with an odd number of vertices of odd degree.

Problem 2. [4 points]
Let \( a_1, a_2, \ldots, a_n \) be \( n \) not necessarily distinct integers. Prove that there is a set of consecutive numbers \( a_k, a_{k+1}, \ldots, a_\ell \) whose sum is divisible by \( n \).

Problem 3. [4 points]
Prove that every set of \( 2^n + 1 \) vectors in \( \mathbb{Z}^n \) (integer coordinates) contains a pair of distinct points whose mean also has integer coordinates.

Problem 4. [4 points]
Prove that for every \( k \geq 2 \) there exists \( n_0 = n_0(k) \) such that every coloring of \( 1, 2, \ldots, n_0 \) in \( k \) colors contains three distinct numbers \( 1 \leq a, b, c \leq n_0 \) that have the same color and satisfy \( a \cdot b = c \).

Problem 5. [6 points]
Prove that as \( n \) tends to infinity, the probability that a random permutation of \( n \) elements does not have a 2-cycle tends to \( e^{-1/2} \).

Problem 6. [6 points]
A transitive tournament is an orientation of a complete graph for which the vertices can be numbered so that \((i, j)\) is a directed edge if and only if \( i < j \).

- Show that every orientation of the complete graph \( K_n \) contains a transitive tournament on \( \lceil \log_2 n \rceil \) vertices.

- Show that if \( k \geq 2 \log_2 n + 2 \), then there is an orientation of \( K_n \) with no transitive tournament on \( k \) vertices.

Problem 7. [6 points]
Let \( g_1(x), \ldots, g_k(x) \) be bounded real functions and \( f(x) \) be another real function. Suppose that there are positive constants \( \epsilon \) and \( \delta \) such that if \( |f(x) - f(y)| > \epsilon \), then \( \max_i(|g_i(x) - g_i(y)|) > \delta \). Prove that \( f \) is also bounded.