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Toward Harmonic Analysis on DAHA
(Integral formulas for canonical traces)

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B. DAHA integrations
C. Canonical traces/forms
D. Rational daha ($A_1$)
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Fourier transform and the Lie theory:

FT $\leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is fine for the classical $F = \int e^{2\lambda x} \{\cdot\} \, dx$ (an outer automorphism of the Heisenberg algebra) and for the Hankel transform. A similar interpretation holds for $F_N = \sum_j e^{\frac{2\pi ji}{N}} \{\cdot\}$ via the Weyl algebra and in arithmetic (Weil, metaplectic representations).

Problems:

1) Cannot be extended to the spherical and hypergeometric functions.

2) A counterpart of $F(e^{-x^2}) = \sqrt{\pi}e^{+x^2}$ at roots of unity is $F_N(e^{x^2}) = \pm \sqrt{Ne^{-x^2}}$; $\pm$ is very important. Weyl algebra doesn’t help.

3) Multi-dim theory. There are 3 ”natural” candidates for FT in $SL(3)$ (reflections in $S_3 \subset SL(3)$); however, FT must be unique:

FT: Polynomials $\mapsto$ Delta-functions.

DAHA: $FT^2$ is essentially id (addresses (1)); FT comes from the transposition of the periods of $E$ with one puncture, which settles (2); any root systems can be managed (3).
A. AHA-DECOMPOSITION

$R \in \mathbb{R}^n$, $W$, $W^a = W \times Q$;

$\mathcal{H} = \langle T_i, 0 \leq i \leq n \rangle / \{ \text{relations} \}:
\{ \text{Coxeter, } (T_i - t^{\frac{1}{2}})(T_i + t^{\frac{1}{2}}) = 0 \}.$

$T_{\hat{w}} = T_{i_l} \cdots T_{i_1}$, $\hat{w} = s_{i_l} \cdots s_{i_1}$,

$l = l(\hat{w})$. $T_{\hat{w}}^* \overset{\text{def}}{=} T_{\hat{w}^{-1}}.$

$< T_{\hat{w}} > = 1(\hat{w} = \text{id})$, 0 otherwise.

$< f, g > \overset{\text{def}}{=} < f^* g > = \sum_{\hat{w} \in W^a} c_{\hat{w}} d_{\hat{w}},$

$f = \sum c_{\hat{w}} T_{\hat{w}}$, $g = \sum d_{\hat{w}} T_{\hat{w}} \in L^2(\mathbb{R}).$

Dixmier: $< f, g > = \int_{\mathcal{H}^\vee} \text{Tr}(\pi(f^* g)) d\nu(\pi).$

SPH: $f, g \in P_+ \mathcal{H} P_+$, $P_+ = \sum_{w \in W} t^{\frac{i(w)}{2}} T_w.$

Macdonald’ case: $\nu_{sph}(\pi), t > 1.$

Arthur, Heckman, Opdam: $0 < t < 1,$

$\int \{ \cdot \} d\nu_{sph}^{an}(\pi) \sum C_{s, S} \int_{s+iS} \{ \cdot \} d\nu_{s, S},$

summed over $s + S = \text{residual subtori}.$

The approach aimed at “reducing” Alg Geometry (Lusztig, ...) in Harmonic Analysis.

DAHA: AHO formula results from “Main thm of q-Calculus”.
B. DAHA INTEGRATIONS

\[
\begin{array}{c|c}
\text{imaginary} \ (|q| \neq 1) & \text{real} \ (|q| \neq 1) \\
\downarrow & \downarrow \\
\text{constant term} \ (\forall q) & \text{Jackson sums} \\
\uparrow & \downarrow \\
\text{the case} \ |q| = 1 = & \Rightarrow \text{roots of unity}
\end{array}
\]

C. CANONICAL TRACES

\(R \in \mathbb{R}^n\) a root system (reduced),
\(W = \langle s_i, 1 \leq i \leq n \rangle\), \(P = \text{weight lattice.}\)

\(\mathcal{H} = \langle X_b, T_w, Y_b \rangle_{q,t}, b \in P, w \in W.\)
Over \(\mathbb{R} \ni q, t, q = \exp(-1/a), a > 0.\)

Generic anti-involution \(\kappa\) of \(\mathcal{H}\):
\(T_w^{\kappa} = T_{w^{-1}}\) and PBW holds:
\(\mathcal{H} \ni H = \sum c_{awb} Y_a^{\kappa} T_w Y_b.\)

DEF. Let \(\{H\}_\varrho^{\varrho} = \sum c_{awb} \varrho(Y_a) \varrho(T_w) \varrho(Y_b)\)
as \(\varrho : \mathbb{R}[Y] \to \mathbb{R} (\text{or } \mathbb{C})\) is a character, and
\(\varrho : \{T_w\} \to \mathbb{R} (\text{or } \mathbb{C})\) is a trace. Then:
\(\{A, B\} \overset{\text{def}}{=} \{A^{\kappa} B\}_\varrho = \{B, A\}.\)
EX. \( \{ H \}_\varphi^{sp} \) for the character \( \varphi : Y_{\omega_i}, T_i \mapsto t^{1/2} \). Then \( \{ A, B \} \) acts through \( \mathbb{R}[X] \times \mathbb{R}[X] \).

EX. \( \varphi : X_b \leftrightarrow Y_b^{-1}, T_w \rightarrow T_{w^{-1}} \) and \( \{ H \}_\varphi^{sp} \); it serves the evaluation, duality formulas.

D. RATIONAL DAHA

\( \mathcal{H}'' \overset{\text{def}}{=} \langle x, y, s \rangle / \text{relations} : \)
\[
[y, x] = 1/2 + ks, \quad s^2 = 1, \\
sxs = -x, \quad sys = -y.
\]

Polynom rep \( \mathbb{R}[x] \):

\( s(x) = -x, \quad x = \text{mult by } x, \quad y \mapsto D/2, \)
\[D = \frac{d}{dx} + \frac{k}{x}(1 - s) \quad \text{(Dunkl)}.\]

REM. \( h := xy + yx \) (Euler operator),
\[
[h, x] = x, \quad [h, y] = -y : \text{osp}(2|1).
\]
\( e := x^2, \quad f := -y^2 : \text{sl}(2). \)

\( x^* = x, \quad y^* = -y, \quad s^* = s \) is special,
serves \( \int f(x)g(x)|x|^{2k} : \text{diverges}; \)

\( \mathbb{R}[x] \) has no \(*\)-form \( (k \not\in -1/2 - \mathbb{Z}_+) \):
\[
\{ 1, y(x^{p+1}) \} = 0 = c_{p+1}\{ 1, x^p \}, \\
c_{2p} = p, \quad c_{2p+1} = p + 1/2 + k.
\]
Replace $y$ by $x + y$, $(x + y)^* = x - y$.

**PBW:** $h = \sum c_{a \epsilon b} \left( (y + x)^* \right)^a s^\epsilon (y + x)^b$.

$\{f\} = \sum c_{o \epsilon o}$, $\{f, g\} = \{f^* g\}$ is through $\mathbb{R}[x] e^{-x^2} \times \mathbb{R}[x] e^{-x^2}$, indeed:

$\mathbb{R}[x] e^{-x^2} = \mathcal{H} / (\mathcal{H}(x + y), \mathcal{H}(s - 1))$.

Let $a, b \in \mathbb{Z}_+$, $p = \frac{a + b}{2}$, then

$\{x^a e^{-x^2}, x^b e^{-x^2}\} = \left( \frac{1}{2} \right)^p \left( \frac{1}{2} + k \right) \cdots \left( \frac{1}{2} + k + p - 1 \right)$.

**Integral formula:** $\{f e^{-x^2}, g e^{-x^2}\} =
\begin{align*}
C \frac{1}{2i} \left( \int_{-i\epsilon + \mathbb{R}} + \int_{i\epsilon + \mathbb{R}} \right) (f g e^{-2x^2} (-x^2)^k) dx,
C = \Gamma(1/2 - k) 2^{-1/2 + k}, \quad k \in \mathbb{C}.
\end{align*}$

**Case** $k > -1/2$. \( < f, g > =
\begin{align*}
= \frac{1}{\Gamma(k+1/2)} \int_\mathbb{R} f g e^{-2x^2} |x|^{2k} dx.
\end{align*}$

**Case** $k = -1/2 - m$. \{f, g\} =

= const \ \text{Res}_0 (f g e^{-2x^2} x^{-2m-1} dx).

**Radical** $= (x^{2m+1} e^{-x^2})$ is unitary w.r.t $\int_\mathbb{R} f g e^{-2x^2} |x|^{-2m-1} dx$.

$\mathbb{R}[x] / (x^{2m+1})$ : module over $\mathfrak{osp}(2|1)$. 

$\mathcal{B}_q \overset{\text{def}}{=} \langle T, X, Y, q^{1/4} \rangle / \text{relations:}$

\[
T X T = X^{-1}, \quad T Y^{-1} T = Y,
\]
\[
Y^{-1} X^{-1} Y X T^2 = q^{-1/2}.
\]

$\mathcal{B}_1 = \pi_{1}^{\text{orb}}(\{E \setminus 0\}/S_2) \cong \pi_1(\{E \times E \setminus \text{diag}\}/S_2)$, $E =$elliptic curve.

$\mathcal{H} \overset{\text{def}}{=} \mathbb{R}[\mathcal{B}_q]/((T - t^{1/2})(T + t^{-1/2})),
\]
$q = \exp(-1/a), \ a > 0, \ t = q^k, \ k \in \mathbb{R}.$

If $t = 1$, $\mathcal{H} = \text{Weyl algebra} \rtimes S_2 : T \mapsto s :$

$s X s = X^{-1}, \ s Y s = Y^{-1},$
\[
Y^{-1} X^{-1} Y X = q^{-1/2}.
\]

\textbf{Fourier transform} $\sim$ automorphism:

$X \mapsto Y^{-1}, \ Y \mapsto T^{-1} X^{-1} T, \ T \mapsto T;$

transposes the periods of $E.$
Relation: \( Y^{-1}X^{-1}YXT^2 = 1. \)
$PSL_2(\mathbb{Z})$ acts projectively

$$
\begin{pmatrix} 11 \\ 01 \end{pmatrix}: Y \mapsto q^{-1/4}XY, \ X \mapsto X, \ T \mapsto T,
\begin{pmatrix} 10 \\ 11 \end{pmatrix}: X \mapsto q^{1/4}XY, \ Y \mapsto Y, \ T \mapsto T.
$$

$X = q^x$. The first: conjugation by $q^{x^2}$.

**Polynom. rep.** $= \text{Ind}(q^{sp\gamma})_{\mathcal{H}_Y}^{\mathcal{H}}$ (PBW):

$\mathcal{L} =$ Laurent polynomials of $X = q^x$,

$$
T \mapsto t^{1/2} s + \frac{t^{1/2} - t^{-1/2}}{q^{2x} - 1} (s - 1),
$$

$Y \mapsto \pi T, \ \pi = sp, \ s f(x) = f(-x),$

$\rho f(x) = f(x + 1/2), \ t = q^k$.

$Y$ is the **difference Dunkl Operator**.

$Y + Y^{-1}$ preserves $\mathcal{L}_{\text{sym}} \overset{\text{def}}{=} \text{sym (even) Laurent polynomials}$.

$Y + Y^{-1} \big|_{\text{sym}}$ is the **$q$-radial part**.
F. INNER PRODUCTS

Macdonald Truncated $\theta$-function: $\mu(x) = \prod_{i=0}^{\infty} \frac{(1-q^i+2x)(1-q^{i+1}-2x)}{(1-q^i+k+2x)(1-q^{i+k+1}-2x)},$

$< f, g >_o \overset{\text{def}}{=} \frac{1}{i} \int_{1/4+P} f(x) T(g)(x) \mu(x) dx,$

$P = [-\pi ia, \pi ia], \ q = \exp(-1/a).$

THM. As $k > -1/2$, it serves $T^* = T$, $Y^* = Y$, $X^* = TXT^{-1}$, which is an anti-involution; therefore, symmetric and positive on $\mathbb{R}[X^{\pm 1}]$.

Proof. a) As $k > -1/2$, $\mu$ has no poles between $\pm \frac{1}{4} + P$; this gives $T^* = T$ (directly, for the path $0 + P$), and also $Y^* = (\pi T)^* = Y$ due to $\pi(\mu) = \mu$ and therefore $\pi^* = T^{-1} \pi T$.

b) $Y(E_n) = q^{-n\#} E_n$, $E_n = X^n + (l.t.),$

$n\# = \frac{n-k}{2}$ as $n \leq 0$, $= \frac{n+k}{2}$ as $n > 0$.

c) $< E_n, E_m >_o = C_n \delta_{nm}$ due to $Y^* = Y$.

d) $C_n = q^{-n\#} \int_{1/4+P} E_n E_n \mu(x) dx > 0$, since $\pi(x) = \bar{x}$ and $\mu(x) > 0$ at $1/4+P$; use $T(E_n) = \pi Y(E_n) = q^{-n\#} \pi(E_n) = q^{-n\#} \bar{E}_n$. $\Box$
Imaginary Integration. On $\mathbb{R}[X^{\pm 1}]$, 

$$< f, g >_\infty \overset{\text{def}}{=} \frac{1}{i} \int_{\frac{1}{4} + i\mathbb{R}} fT(g)q^{-x^2} \mu(x)dx$$

$$= \frac{1}{i} \int_{\frac{1}{4} + \mathbb{P}} fT(g) \sum_{j=-\infty}^{\infty} q^{n^2/4 + nx} \mu(x)dx$$

is a symmetric and positive form, serving $T^\zeta = T$, $X^\zeta = TXT^{-1}$, $Y^\zeta = q^{-1/4}XY$.

G. ANALYTIC CONTINUATION
The anti-involution $\zeta$ is generic:

$$\varrho(\sum_{\alpha, \beta \in \mathbb{Z}} c_{\alpha \beta}(Y^\zeta)^{\alpha \cdot \epsilon \cdot \beta} \overset{\text{def}}{=} \sum c_{\alpha \beta} t^{\alpha + \epsilon + \beta} , \{ A, B \}^\zeta = \varrho(A^\zeta B) = \{ B, A \}^\zeta \text{ on } \mathcal{H}.$$  
It acts via $\mathcal{L} \times \mathcal{L}$, $\mathcal{L} = \mathbb{R}[X^{\pm 1}]$, and $\{ 1, 1 \} = 1$.  
This form is analytic for all $k \in \mathbb{C}$.

THM. $G(k) \{ f, g \}^\zeta = < f, g >_\infty$,  

$$G(k) = \sqrt{\pi a} \prod_{j=1}^{\infty} \frac{1 - q^{k+j}}{1 - q^{2k+j}}, \Re k > -1/2.$$  

Proof. Generally, let $C = \{ \epsilon + i\mathbb{R} \}$ and  

$$\Phi_k^\epsilon(f, g) \overset{\text{def}}{=} \frac{1}{i} \int_{\epsilon + i\mathbb{R}} fT(g)q^{-x^2} \mu(x)dx.$$  

Bad $k = \{ 2C - 1 - \mathbb{Z}_+, -2C - \mathbb{Z}_+ \}$;  
so $\{ \Re k > -1/2 \}$ are good as $\epsilon = -1/4$.  

Case $\epsilon = 0$. $\Phi_0^k(f, g)$ gives $G(k)\{ f, g \}_{\epsilon}^0$ for $\Re k > 0$ only!

However it is symmetric for all $k$ and $T^\epsilon = T$, $X^\epsilon = TXT^{-1}$.

Relation: $\Phi^k_{1/4} = \Phi_0^k + A(-k) \Pi(-k) F(-k)$,

$A(\tilde{k}) = \frac{\sqrt{\pi a}}{2} \sum_{m=-\infty}^{\infty} q^{m^2+m\tilde{k}}$,

$\Pi(\tilde{k}) = \prod_{j=0}^{\infty} \frac{(1-q^{\tilde{k}+j})(1-q^{\tilde{k}+j+1})}{(1-q^{1+j})(1-q^{2\tilde{k}+j+1})}$,

$F(\tilde{k}) = fT(g)(x \mapsto \tilde{k}/2)$.

Here $\Phi_0^k + A(-k) \Pi(-k) F(-k)$ is analytic and therefore coincides with $G(k)\{ f, g \}_{\epsilon}^0$ as $\Re k > -1$. It is also symmetric for ALL $k$:

$fT(g)(-\frac{k}{2}) = t^{1/2} fg(-\frac{k}{2}) = T(f)g(-\frac{k}{2})$.

since $T = \frac{q^{2x+k/2}-q^{-k/2}}{q^{2x-1}} s - \frac{q^{k/2}-q^{-k/2}}{q^{2x-1}}$,

where $(q^{2x+k/2} - q^{-k/2})(x \mapsto -k/2) = 0$.

MAIN THM. $G(k)\{ f, g \}_{\epsilon}^0 = \Phi_0^k + \sum_{\tilde{k} \in \tilde{K}} A(\tilde{k}) \Pi(\tilde{k}) F(\tilde{k})$ as $\Re k < 0$,

$\tilde{K} = \{-k\} \cup \{-k-j, k+j\}$ for $j = 1, \ldots, \lfloor \Re(-k) \rfloor$; \[ \lfloor \cdot \rfloor = \text{integer part}. \]

Note. $\tilde{K}$ must be symmetric but at $-k$. 
COR. \( \{ f, g \}_{\kappa}^0 \) is degenerate exactly at the poles of \( G(k) : k = \frac{-1/2 - m}{}, m \in \mathbb{Z}_+ \).

Then \( \widetilde{K}/2 \) is \( \pi \)-invariant and

\[
\text{Funct}(\widetilde{K}/2) = \mathcal{L}/\text{Radical}\{,\} \quad \text{is an } \mathcal{H} \text{-module of dim } 2m + 1.
\]

The radical is unitary with respect to \( \Phi_0^k \).

Rational Limit: \( q = e^h, \ h \to 0, \)
\[ Y = e^{-\sqrt{h}y}, \ X = e^{\sqrt{h}x}. \]

Rational DAHA\(= \lim \mathcal{H} \).

**H. P-ADIC LIMIT**

**GENERAL THEOREM.** For generic \( \kappa \), the canonical trace/form is a sum of integrals over affine residual subtori.

As \( q \to \infty \) (renoramlization is needed), the Fourier transform of \( \mathcal{L} \) tends to the spherical part of the affine module \( \mathcal{H} \).

**Only non-affine residual subtori contribute.**

The radical filtration (small \( \Re k < 0 \)) becomes a filtration of \( \mathcal{H} \)-modules (!).
I. JANTZEN FILTRATIONS

INGREDIENTS: a) a PBW-generic anti-involution $\kappa$ of $\mathcal{H}$ (w.r.t. the subalgebra $Y$),
   b) the canonical trace associated with $\kappa$ (requires a character $\varrho$ on $Y$ and the trace of the non-affine Hecke algebra),
   c) the “Shapovalov form”, a combination of (a) and (b) with $\langle 1, 1 \rangle = 1$ ($k$-analytic).

PROBLEM: “expand” the canonical trace and the Shapovalov form $\{f, g\}_\varrho^\kappa$ as a sum of integrals over the affine residual subtori.

Analytic Jantzen filtration of $\mathcal{L}$ (AHA (!) but not DAHA modules) for $\Re k < 0$. The top/first term is the sum over the smallest tori, the bottom/last term is the integration over $i\mathbb{R}^n$.

Algebraic Jantzen filtration of $\mathcal{L}$ is in terms of DAHA (!) modules at singular $k_o$ w.r.t. $\tilde{k} = k - k_o$. The top form is the coefficient of $\tilde{k}^0$, the bottom term is (in known examples) the straight integration over $i\mathbb{R}^n$. 
Now restrict the $m$-th form to the radical of the $(m - 1)$-th form and consider its radical, continue. The construction gives the total irreducible decomposition of $\mathcal{L}$ for $A_n$ ("Kasatani conjecture" proven via the rational limit).

**J. RATIONAL CASE**

Represent $\{f, g\}^0_\omega$ as an integral over the boundary of the tube neighborhood of the resolution of the cross $\prod x_\alpha = 0$ extended to $\infty$, e.g., over $\pm i\epsilon + \mathbb{R}$ for $A_1$. Resolution: Ch, de Concini - Procesi, Beilinson-Ginzburg.

**Conjecturally**: singular $k_\omega$ are the $k$ when this integral can be reduced to integrations "over smaller subtori" (e.g., over a compact integration counter in case of the point).

**Conjecturally**: for singular $k_\omega = s/d_i$, the bottom module is "semi-simple" (à la Suzuki for $A_n$). It may be always unitary for $s = 1$ with respect to the "pure" $\int_{\mathbb{R}^n} \{\cdot\} e^{-x^2} \, dx$ (as for $A_1$; Etingof and his students for $A_n$).

A relation to singularities à la Shokurov.