Khovanov's homology for tangles and cobordisms
(Khovanov's homology and TQFT).

1. Recall that Khovanov's homology associates to a knot (link) a chain complex of vector spaces. Its Euler characteristic is Jones' polynomial.

2. Let us review this construction (this is really an interpretation of the picture we had before.

   ![Diagram]

   • Trefoil knot. \[ \square \] is the Khovanov bracket (certain chain complex)
   • Crossings, \[ \cross \times \square \], we have \[ N_+ = 0 \]
   • Cube: vertices are pictures of smoothings \[ X \to \square \](\( \square \)) labelled by sequences of 0's and 1's.
     The height of vertex \( x \) = sum of coordinates
     Vertices of height \( k \) project on \( k - N_+ \) (\(-N_-\) shift).
     (In general one replaces this picture by an \( n \)-dimensional cube

   - Edges are cobordisms, i.e. oriented 2-dim surfaces
     embedded in \( \mathbb{R}^2 \times [0,1] \) with boundary in \( \{0,1\} \). whose
top and bottom boundaries are corresponding smoothings.

   - View from the top.

   - Signs are chosen in such a way
     that each face has odd number of \(-\) signs
     Explicitly if \( \xi = (\xi_1, \xi_2, \xi_3) \) and \( \xi_j = * \) then
     \[ \text{sign} = (-1)^{\sum \xi_j} \]
3. This picture can be generalized for 

By definition, a tangle is a knot with possibly loose ends. More precisely, this is a part of knot/link diagram bounded by a disk.

Cobordisms are now bounded inside cylinder.

4. Idea: Interpret this cube as a complex [\text{\text{[I]}}, \text{in the sense of homological algebra}], by thinking of smoothings as spaces and cobordisms as maps.

We'll set: [\text{\text{[I]}}, \text{r-th chain space "spaces" (smoothings)}] at height r.

The differential will be the "sum" of given cobordisms.

We can regard cobordisms as maps between smoothings and compose them.

5. Linearization:

Consider vector spaces consisting of formal linear combinations of smoothings (i.e. things like a \( \cdot \) + b \( \cdot \) + c \( \cdot \) etc.), and linear maps are "matrices of cobordisms."
Such linear maps can be composed via usual matrix multiplication (placing a cobordism on top of another). Our picture for a trefoil knot can be interpreted as a chain of maps

\[
[K] = ([K]^{-3} \rightarrow [K]^{-2} \rightarrow [K]^{-1} \rightarrow [K]^0) \quad \text{in the above category. Call it } \text{Cob}^3
\]

Each arrow = sum of cobordisms.

* We mod out lin. transformations in \text{Cob}^3 by the following relations:

(S) if a cobordism contains closed sphere, it is set 0.

\[
\text{Example: } \begin{array}{c}
\includegraphics[width=1cm]{sphere} \\
\end{array} = 0
\]

(T) if a cobordism contains a torus, it is replaced by a factor of 2.

\[
\text{Example: } \begin{array}{c}
\includegraphics[width=1cm]{torus} \\
\end{array} = 2
\]

(4Tubes) If given a cobordism \( C \) its intersection with some ball is a union of 4 disks \( D_1, D_2, D_3, D_4 \).

Let \( C_{ij} = C - D_i - D_j \) + tube connecting boundaries \( i, j \)

Then \( C_{12} + C_{34} = C_{13} + C_{24} \)

(i.e. morphisms set = 0).

Let \( \text{Cob}^3 \) denote the resulting category.

6) **Complex**: We define maps \( d^r : T^r \rightarrow T^r \) as the sum of all cobordisms. We claim that \( d^r \circ d^{r-1} = 0 \), i.e. that

\[
T^r \rightarrow T^{r+1} \rightarrow T^r
\]

is complex.

**Pf**: Each face contains odd number of "-" signs. So it suffices to check each face commutes (without signs). Reason: spatially separated "saddles" can be raised or lowered independently.
The complex $T$ itself is not an invariant. The invariant is the homotopy class of $T$

**Def.** Let

\[
\begin{array}{ccccccccc}
A^{r-1} & \xrightarrow{d^r} & A^r & \xrightarrow{d^r} & A^{r+1} \\
F^r & \xrightarrow{h^r} & C^r & \xrightarrow{h^r} & F^{r+1} \\
\downarrow F^r & & \downarrow C^r & & \downarrow F^{r+1} \\
B^{r-1} & \xrightarrow{d^r} & B^r & \xrightarrow{d^r} & B^{r+1} \\
\end{array}
\]

be complexes. Let $F, G : A \to B$ be two chain maps.

We say that $F, G$ are homotopic if there exists a "backwards diagonal" $h^r : A^r \to B^{r-1}$ so that

\[F^r - G^r = h^r \circ d^r + d^{r-1} \circ h^r\]

We say $A, B$ are homotopic if there exist chain maps $F_1 : A \to B$, $F_2 : B \to A$ s.t. $F_1 F_2$, $F_2 F_1$ are homotopic to id.

**Remark.** Homotopic complexes have the same homology group.

**Main Theorem.** The homotopy class of $[T]$ regarded in $\text{Cob}_3^{n}$ (where $n \geq 6$) is an invariant of $T$. (I.e., it is independent under ordering of layers in a cube and is invariant under Reidemeister moves.)

**7. Functoriality.**

We associate (classes of) chain complexes to links/tangles.

Note that there are maps between chain complexes (chain maps).

Are there "maps" between tangles (links)?

For un-links and tangles without crossings we have cobordisms, what about non-trivial links/tangles?

In general, one considers 2 tangles oriented surfaces embedded in $\mathbb{R}^3$. They can be regarded as "maps between knots/links/tangles."

One can show that $T \mapsto [T]$ extends to maps (i.e., is a functor) i.e. to maps between tangles one can assign chain maps.
Figure 1: The main picture. See the narrative in Section 2.