Knot Theory - Complement of Figure-8 knot

**Figure-8 knot:**

- **Basic idea:**
  - Using Reidemeister moves, write the figure-8 knot in a different way, and annotate this picture.
  - Then, we can view the result as the gluing of four 2-cells, which gives us a "surface" such that the knot is embedded in the boundary.
  - Analyze the "inside" and "outside" of this object, and consider what we get when we remove the knot from the boundary.
  - Finally, we "glue" the inside and outside objects together. Is this a manifold?

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First, re-draw Fig. 8-knot inserting two arrows, 1 red, 1 green as follows:
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To get a better idea of this, imagine that this object has a surface, so that the above picture almost appears as a tetrahedron. Then, we can "unfold" the above picture, or view it as a "gluing" of four two-cells.
**Definition:** An n-dimensional gluing consists of a finite set of n-simplices, a choice of pairs of simplicial maps such that each 'map' appears in exactly one of the pairs, and an affine identification map between the maps of each pair.

So, we want a collection of 4 two-simplices such that one red arrow and one green arrow appear on each:

We can view each side of the previous diagram as a face with a strip and a twisted tube:

![Diagram](image)

*Note:* Each face contains one red arrow and one green arrow; apply this to the definition above.

Also, note that we have "untwisted" the base of these faces:

![Diagram](image)

So, gluing these cells together gives us the object we have on the previous page.

*Also, this object is orientable, and so there is a clear interior/interior.*

Then, we can consider the span on the interior and the exterior.

To do this, begin with a "balloon" (see figure), on the interior of our surface.

Suppose that the knot is covered with fresh black paint, and the red and green arrows have also been freshly painted. As we inflate the balloon, the surface of the balloon will press against the boundary of our surface, and the paint will leave an impression, as follows:
This is the imprint left on our 'bulbous' by the point, and represents the boundary of the space on the interior of our surface.

But, the black lines represent the knot, and since we are interested in the complement, we wish to shrink the knot to zero and analyze the result: Shown in 2 steps, we have:

And finally, a tetrahedron: \( T \).

Similarly, we can inflate a balloon on the exterior of the surface in just the same way, leading to the figure: (also a tetrahedron \( T' \))

Now, we have tetrahedra that represent the inside \( (T) \) and outside \( (T') \) of the surface, and the figure \& knot has been (almost) removed.
Next, we must 'glue' the two tetrahedra together, which amounts to identifying red arrows of $T$ with red arrows of $T'$. We can label $T, T'$ in the following way to make this more intuitive:

To make the appropriate glue, we must identify:

$$
\begin{align*}
A & \to A' \\
B & \to B' \\
C & \to C' \\
D & \to D'
\end{align*}
$$

Next, we wish to analyze the kind of object we have left after completing this glue, and determine if it is a manifold.

To do this, we can follow the orbit of one of the vertices:

Beginning with $V_1$, we see this is a vertex of $A$, and by the glue, we get:

$$V_1 = V_1'$$

Next, consider it as a vertex of $B'$, $\to B \Rightarrow$

$$V_1 = V_1' = V_4 \text{ etc...}$$

Continuing on like this, we see that all vertices will be identified, being only on a vertex.

Similarly, red edges and green edges will be identified with a single red edge and a single green edge.

But, is this a manifold?
Proposition: Let $X$ be a simplicial complex of dimension 3. Then, $X$ is a manifold $\Rightarrow$ the link of any vertex is homeomorphic to $S^2$.

We can calculate the link of our vertex explicitly by unfolding our graph and calculating the Euler characteristic of the link $L$:

\[ (-T) \quad (T') \]

To calculate the link, we calculate the Euler characteristic after gluing: $V - E + F$.

First, it is clear that each glue does not result in the identification of any faces with each other. So, $F = 8$.

Next, each time we glue (say $A \rightarrow A'$), one edge will be identified with precisely one other edge (so we lose half of them). Thus, $E = 24/2 = 12$.

Finally, to count the number of vertices, we can follow the orbit of a vertex. Beginning with $V_1$:

\[ \begin{align*}
A &\rightarrow A' \Rightarrow V_1 \rightarrow V_2, \\
C &\rightarrow C' \Rightarrow V_3 \rightarrow V_5, \\
B &\rightarrow B' \Rightarrow V_3 \rightarrow V_4, \\
A' &\rightarrow A \Rightarrow V_4 \rightarrow V_5, \\
D &\rightarrow D' \Rightarrow V_5 \rightarrow V_6, \\
B' &\rightarrow B \Rightarrow V_6 \rightarrow V_1.
\end{align*} \]

So, in this gluing pattern, each vertex is identified with 6 other vertices.

Thus, $V = 24/6 = 4$.

So, the link of $V$ has Euler characteristic 0, and so is a Torus, which is not homeomorphic to $S^2$. Thus, this is not a manifold.
However, recall that the figure 8 knot passed through the tip and point of our arrows in the first diagram, and these are precisely the vertices in question. Thus, we can delete the vertex and analyze the remaining object.

It turns out if we compute the Euler characteristic of the object with no vertex, the calculation is identical. So, we have that the complement of the figure 8 knot is precisely a Solid Torus.