Overview

My research interests range between algebraic $K$-theory and equivariant homotopy theory. I am currently extending the trace methods of [16],[30],[49] to the Real algebraic $K$-theory of rings and ring spectra with Wall antistructures of [39], in parallel with an equivariant theory of Goodwillie calculus of functors [35]. My current main objective is to prove a Dundas-Goodwillie-McCarthy theorem for Real and Hermitian algebraic $K$-theory, as well as relating equivariant calculus to a theory of equivariant operads. The interplay between Real $K$-theory and equivariant calculus should lead to calculations in Hermitian $K$-theory of rings [43] and in Hermitian $A$-theory [55], and on the long term it should find important applications in geometric topology and in surgery theory.

The document is divided into two sections, one on $K$-theory and one on calculus. Each section is divided into a background part and a part specific to my research project. The research goals are divided into short and longer term goals. The short term goals are the main problems that I am currently investigating, and that I believe they could be solved within one or two years from now. The longer term goals are bigger problems which arise naturally from the short term goals.

1 Real $K$-theory and traces

1.1 Background

Higher algebraic $K$-theory was introduced in the seminal work of [51]. It is a construction that associates to a ring $A$ a space (or spectrum) $K(A)$ which is a topological group completion for a moduli space of finitely generated projective modules over $A$. This theory was later extended in [56] to ring spectra, and in fact to a more general input known today as Waldhausen categories. The homotopy groups of the $K$-theory of a Waldhausen category appear in various fields of mathematics: in algebraic geometry and in number theory where the input is the category of perfect modules over a scheme or the ring of integers in a number field ([54] and [59]), in algebra where the input is a group-ring ([4]), and in geometric topology where the input is the category of finite retractive $CW$-complexes over a space or the spherical group-ring of a loop space ([56], [57] and [32]). In each of these cases algebraic $K$-theory is related to important open problems of arithmetic and geometric nature.

Not surprisingly, such an interesting invariant turns out to be extremely difficult to calculate. Even the algebraic $K$-theory of the ring of integers is not quite completely understood (it is up to the Kummer-Vandiver conjecture). A successful strategy for computations is to compare algebraic $K$-theory to objects of a more homotopy-theoretical nature where the computations are somewhat more approachable. These objects are typically Bökstedt’s topological Hochschild homology THH of [17] or the topological cyclic homology TC of [16]. One successful comparison tool is the Dundas-Goodwillie-McCarthy Theorem of [49], [28] and [29] which compares $K$-theory and topological cyclic homology via the trace map $Tr: K \to TC$ of [16]. This theorem states that for suitable maps of ring spectra $A \to B$ the corresponding square
relating $K$-theory and TC

$$
\begin{array}{ccc}
K(A) & \overset{\text{Tr}}{\longrightarrow} & TC(A) \\
\downarrow & & \downarrow \\
K(B) & \overset{\text{Tr}}{\longrightarrow} & TC(B)
\end{array}
$$

is homotopy cartesian after completion at a prime. These “trace methods” led to several computations, for example of the $K$-theory of perfect fields in [38], of truncated polynomial algebras in [37] and [1], or of the sphere spectrum in [44],[52] and [13].

Hermitian $K$-theory is a similar invariant, where the input ring is replaced by a ring with a Wall antistructure [58] (e.g. a commutative ring). It was first defined in [43] as a topological group-completion of a moduli space of Hermitian forms. Similar to algebraic $K$-theory, Hermitian $K$-theory has great arithmetic and geometric relevance. The Hermitian $K$-theory of an Abelian number field is related to some special values of the Dedekind zeta-function ([9]), and the Hermitian $K$-theory of an integral group-ring is related (up to quadratic refinement) to the surgery exact sequence. Karoubi’s definition was later extended to exact categories with duality ([41] and [53]) and to a Hermitian $K$-theory of spaces in [55], again with important relations to geometric topology and surgery ([60],[61] and [62]).

Also in the Hermitian case the calculations are hardly accessible, and a theory of traces is still lacking.

1.2 The research project

The starting point of my research is Hesselholt and Madsen’s recently developed theory of Real $K$-theory [39]. From an exact category with weak equivalences $\mathcal{C}$ and duality $D: \mathcal{C}^{\text{op}} \to \mathcal{C}$, they construct a $\mathbb{Z}/2$-equivariant spectrum $KR(\mathcal{C})$, whose underlying spectrum is the $K$-theory $K(\mathcal{C})$ and whose fixed points spectrum is the Hermitian $K$-theory of $\mathcal{C}$

$$
KR(\mathcal{C})^{\mathbb{Z}/2} \simeq KH(\mathcal{C})
$$

This improves earlier constructions of [53] and [62] where a similar Real $K$-theory was constructed as a $\mathbb{Z}/2$-space. Having a theory valued in stable $\mathbb{Z}/2$-equivariant homotopy theory is a main prerequisite for a theory of traces.

In my PhD thesis [20] I developed a $\mathbb{Z}/2$-equivariant analogue of THH for spectral categories with duality: the Real topological Hochschild homology THR. Its $\mathbb{Z}/2$-fixed points spectrum plays the role of a Hermitian version of THH. I showed in [20] that THR enjoys structural properties which are analogous to the classical theory of [31] and, for linear categories, I constructed a $\mathbb{Z}/2$-equivariant trace map $KR \to THR$. In particular this provides a natural map from Hermitian $K$-theory

$$
KH(A) \rightarrow THR(A)^{\mathbb{Z}/2}
$$

for every (discrete) ring with Wall-antistructure $A$. The spectrum THR has a natural $S^{1,1} \times \mathbb{Z}/2$-action, where $S^{1,1}$ is the sign-representation sphere. The Real topological cyclic homology TCR is a $\mathbb{Z}/2$-spectrum defined from the action on THR of the finite subgroups of $S^{1}$. 

2
**Short term goal 1.** Find a convenient collection of maps of rings with antistructure $f: A \to B$ for which the square of $\mathbb{Z}/2$-spectra

\[
\begin{array}{ccc}
KR(A) & \xrightarrow{\text{Tr}} & TCR(A) \\
\downarrow f_* & & \downarrow f_* \\
KR(B) & \xrightarrow{\text{Tr}} & TCR(B)
\end{array}
\]

is homotopy cartesian after suitable completion.

**Progress.** I am following the strategy of [49], of proving the result for “split square-zero extensions” $A \ltimes M \to A$ first, and then enlarging the class of maps by an algebraic inductive argument on the nilpotence of the kernel.

In the split square-zero extensions case, the main result of my PhD thesis [20] shows that the trace map defines an equivalence between a suitable stabilization of $KR$ and $THR$. This is the first place where the equivariant calculus of functors of §2 intertwines with Real $K$-theory, as this stabilization arises as a $\mathbb{Z}/2$-equivariant derivative of the functor $K(A \ltimes M(-))$. This is analogous to the main result of [30], which constitutes the first step of the proof of [49].

The next step is to show that $THR$ is also the $\mathbb{Z}/2$-equivariant derivative of $TCR$. This should use methods which are similar to [36], and partial results are already included in my thesis. This would prove that the square in question is homotopy cartesian after taking $\mathbb{Z}/2$-derivatives. The final step is to use equivariant calculus of functors to remove the derivatives. I developed a calculus machinery sufficient for this procedure in [21], together with the necessary analytical properties of Real $K$-theory. The corresponding analytical properties of $TCR$ still need to be explored.

The Real $K$-theory of [39] is defined for exact categories with duality. A similar theory had already been established in [62] for a larger class of Waldhausen categories “with Spanier-Whitehead duality”. This theory was constructed to model the Hermitian $K$-theory of spaces of [55]. The construction of [62] solves some very technical issues about the non-functoriality of the Spanier-Whitehead dual of a space, but it is rather indirect and it does not seem to produce a $\mathbb{Z}/2$-equivariant spectrum.

**Short term goal 2.** Extend the construction of Real $K$-theory to a convenient class of “Waldhausen categories with duality”. This class of categories should include the category of finite modules over a ring spectrum $A$ with the function spectrum “duality” $F_A(-, A)$, and support a trace map $KR(A) \to TCR(A)$ of $\mathbb{Z}/2$-equivariant spectra.

**Progress.** I am combining the constructions of [62] and [39]. The central idea is to define a Waldhausen structure on the strictification construction “$xC$” of [62]. There are many technical issues related to the existence of structures (pullbacks, duality,...) only “up to equivalence”. When set it up correctly, these issues should be taken care of by a mix of the $S^{2,1}_*$-construction of [39] and Blumberg and Mandell’s homotopical $S_*$-construction of [14].

**Longer term goal 3.** Study involutions on Witt vectors and generalize the computations of [38] of the $K$-theory of the dual numbers of perfect fields to the Real context.
Strategy. The strategy to follow would be the one of [38], using the Real Dundas-McCarthy theorem above. It involves a calculation of the homotopy groups of the $\mathbb{Z}/2$-fixed points of $\text{TCR}(k[e]/e^2)$ in terms of the ring of Witt vectors $W(k)$. An intermediate stage is the fixed-points analogue of the isomorphism of [38] between $\pi_s \text{THH}(k)^{C_p^n}$ and the symmetric algebra $S_{W_{n+1}}\{\sigma\}$, where $W_{n+1}$ is the ring of Witt-vectors of length $n+1$ and $\sigma$ is a generator of degree 2.

**Longer term goal 4.** Extend the Real Dundas-Goodwillie-McCarthy Theorem to ring spectra with antistructures.

Strategy. This would either involve an approximation of ring spectra with antistructure as limits of simplicial rings with antistructure similar to [28], or a “spectral proof” of the Dundas-Goodwillie-McCarthy theorem for suitable radical extensions of ring spectra with antistructures. The second approach involves an version of the theorem for the $K$-theory of exact infinity-categories, which is work in progress in a joint project with Clark Barwick [6].

An immediate consequence of the extension of Real $K$-theory to ring spectra is the existence of a $\mathbb{Z}/2$-spectrum $\text{KR}(S_{\mathbb{Z}/2}^G \wedge G_+)$, where $S_{\mathbb{Z}/2}$ is the $\mathbb{Z}/2$-equivariant sphere spectrum and $G$ is a topological group with the anti-involution defined by inversion. When $G = \Omega X$ is a loop space, the underlying spectrum is equivalent to the $A$-theory spectrum $A(X)$, and the $\mathbb{Z}/2$-fixed-points should be equivalent to Vogell’s Hermitian $A$-theory $AH(X)$ of [55]. Classically $A(X)$ splits as

$$A(X) \simeq (S \wedge X_+) \vee Wh(X)$$

where $Wh(X)$ is the Whitehead spectrum. When $X = M$ is a manifold $Wh(M)$ is a geometric object related to a stabilization of the space of $h$-cobordisms of $M$ ([57]). Moreover the splitting is realized through the trace map $K(S \wedge G_+) \to \text{THH}(S \wedge G_+)$. The spectrum $Wh(M)$ has a natural involution whose homotopy orbits are related to the diffeomorphism group of $M$, through surgery theory ([60] and [61]). Thus an equivariant understanding of the splitting of $A(M)$ could lead to calculations of the diffeomorphism group of $M$.

**Longer term goal 5.** See if the Real trace map extends the splitting $A(X) \simeq (S \wedge X_+) \vee Wh(X)$ to an equivariant splitting

$$\text{KR}(S_{\mathbb{Z}/2}^G \wedge \Omega X_+) \simeq (S_{\mathbb{Z}/2} \wedge X_+) \vee RWh(X)$$

where $RWh(X)$ is the Whitehead spectrum with the involution of [60] and [61]. On fixed-points, this would induce a splitting of Vogell’s Hermitian $A$-theory $AH(X)$.

Strategy. This is an ambitious goal. The proof of [57] is mostly categorical, and so is the construction of the involution on KR described in the short term Goal 2, giving a chance to the arguments of [57] to be extended to the Real context.

**Longer term goal 6.** Use the Real trace and the assembly map in THR to obtain information about the Hermitian assembly map $\text{KH}(\mathbb{Z}) \wedge BG_+ \to \text{KH}(\mathbb{Z}[G])$.
Strategy. A successful strategy for the study of the K-theoretic assembly map is to study the assembly of spherical group-rings $K(S) \wedge BG_+ \to K(S[G])$, as in [15]. Similarly, the Real K-theory assembly $KR(S_{\mathbb{Z}/2}) \wedge BG_+ \to KR(S[G])$ and its THR counterpart could retain important information about the integral Hermitian assembly map, hopefully rational injectivity. These kind of results are highly relevant in geometric topology [45]. This is work in progress with Crichton Ogle [26].

2 Equivariant calculus of functors

2.1 Background

Calculus of functors was first introduced by Goodwillie in [33] and [34] with a primary focus on algebraic K-theory of spaces. It was later refined with a broader perspective in [35] by the same author, and it was extended to a more abstract homotopy theoretical framework in [10] and [11]. The main idea of this theory is to construct, from a homotopy invariant functor $F: \mathcal{C} \to \mathcal{D}$ between two homotopy theories $\mathcal{C}$ and $\mathcal{D}$ (typically pointed spaces or spectra), a tower of functors

$$
\cdots \to P_{n+1}F \to P_nF \to \cdots \to P_1F
$$

that mimics the Taylor expansion of a real-valued function. The functor $P_nF: \mathcal{C} \to \mathcal{D}$ is “the best approximation of $F$ by an $n$-homology theory”, and it is called the $n$-excisive approximation of $F$. The $n$-excision property is defined in terms of the behavior of a functor on certain cubical diagrams in $\mathcal{C}$ of dimension $n+1$. In particular $P_1F$ is a homology theory, in the sense that it sends homotopy pushout squares in $\mathcal{C}$ to homotopy pullback squares in $\mathcal{D}$.

There are two fundamental theorems that are proved in [35]. The first finds a technical condition on a functor on pointed spaces $F: \text{Top}_* \to \text{Top}_*$ that insures that the tower above converges to $F$

$$
F(X) \xrightarrow{\approx} \text{holim}_n P_nF(X)
$$

for spaces $X$ which are sufficiently highly-connected. Examples of functors that satisfy this condition include the identity functor, the algebraic K-theory of spaces functor $A(-)$, the relative algebraic K-theory functor $\tilde{K}(A \times M(-))$ from Goal 1, and the stable mapping space functor $\Omega^\infty \Sigma^\infty \text{Map}_s(K, -)$ out of a finite pointed CW-complex $K$. The second fundamental result is the characterization of the layers $D_nF = \text{hofib}(P_nF \to P_{n-1}F)$ of the tower above, as functors of the form

$$
D_nF(X) \simeq (C_F \wedge X^{\Sigma_n})_{h\Sigma_n}
$$

for some spectrum $C_F$ with a naïve $\Sigma_n$-action. Here $\Sigma_n$ acts on the smash powers of $X$ by permuting the factors, and $(-)_{h\Sigma_n}$ denotes the homotopy orbits.

This theory has a big impact in algebraic K-theory and in unstable homotopy theory. It is applied in [49] to the algebraic K-theory and to the topological cyclic homology functors to prove the Dundas-Goodwillie-McCarthy Theorem from §1.1. The trace map in A-theory $A(X) \to \text{THH}(S \wedge \Omega X_+)$ is equivalent to the first excisive approximation map $A(X) \to P_1A(X)$, and it is used in [56] to construct the $A$-theory splitting of §1.2. The layers of the
tower of the identity functors are described in [42], and they are used in [3] to calculate the periodic homotopy of odd-dimensional spheres.

Now suppose that $G$ is a finite group an that $\mathcal{C}^G \to \mathcal{D}^G$ is a functor between $G$-equivariant homotopy theories (typically the categories of pointed $G$-spaces or of genuine $G$-spectra). Standard Goodwillie calculus can certainly be applied to $F$, but not without disappointment. For example the layer $D_n F$ of a homotopy invariant functor $F$: $\text{Top}^G \to \text{Top}^G$ is classified by a spectrum in the category $\text{Top}^G$ (with a $\Sigma_n$-action), that is to say a naive $G$-spectrum. What we would like is a theory where the layers of the tower are genuine $G$-spectra with some sort of $\Sigma_n$-action. In particular if $F$ sends the point to the point there is an equivalence

$$P_1 F(X) \simeq \hocolim_n \Omega^n F(\Sigma^n X)$$

but a “genuine theory” would require a stabilization of the form $\hocolim_V \Omega^V F(\Sigma^V X)$ where $V$ runs through the finite dimensional representations of $G$.

2.2 The research project

In spent the last couple of years researching on this subject, resulting in four papers [21], [22], [25] and [23]. The first paper studies the interaction between equivariant calculus and Real algebraic $K$-theory of rings that is needed for the Real Dundas-Goodwillie-McCarthy Theorem of Goal 1. The other papers develop a theory of equivariant calculus based on $G$-indexed cubes.

The central idea for developing this theory is to replace the $(n+1)$-cubes used in [35] to define $n$-excision with cubes which are indexed on finite $G$-sets. This requires a theory of “genuine” equivariant homotopy limits and colimits for diagrams which are indexed on a category with a $G$-action, and the very notion of an “equivariant model category”. These foundations have been laid out in [25] in a joint project with Kristian Moi. In [23] I develop a notion of higher order equivariant excision for functors $F$: $\mathcal{C}^G \to \mathcal{D}^G$ between equivariant homotopy theories, and an equivariant version of the Goodwillie tower which is indexed on finite $G$-sets. In particular by considering only the free $G$-sets this gives a tower

$$\cdots \to P_{(n+1)G} F \to P_{nG} F \to \cdots \to P_G F$$

where $P_G F$ is a genuine equivariant homology theory. Among other results the paper contains a calculation of the layers of this tower for the identity functor on pointed $G$-spaces in terms of the partition complexes, analogous to [42] and [3]. There are many angles and developments of the theory that still need to be investigated.

**Short term goal 7.** Calculate the equivariant tower of the Real $K$-theory functor $\widetilde{KR}(A \ltimes M(-))$: $\text{Top}^{Z/2}_{*} \to \text{Sp}^{Z/2}_{*}$ and of the equivariant $A$-theory functor $A_G$: $\text{Top}^G \to \text{Sp}^G$ of [50].

**Progress.** I proved in my thesis and in [21] that the first stage $P_{Z/2}\widetilde{KR}(A \ltimes M(-))$ of the equivariant tower of the Real $K$-theory of a split square zero extension $A \ltimes M \to A$ is equivalent to the Real topological Hochschild homology functor $\text{THR}(A; M(S^{1,1} \wedge (-)))$. The higher stages should be described in terms of a Real TR-theory with coefficients, analogous to the work of [48].
The equivariant $A$-theory $A_G$ of [50] is a functor whose fixed-points $A_G(X)^H$ by a subgroup $H$ of $G$ is the $K$-theory of the category of finite retractive $H$-spaces over $X$. In a joint project with Mona Merling [24] we are trying to describe the equivariant deloopings of $A_G$ in terms of iterations of the $S_\bullet$-construction indexed on $G$-sets. This would give a better description of $A_G$, and it is a starting point for understanding the layers of the equivariant tower of $A_G$. These should be described in terms of equivariant loop spaces of the smash powers of $X$.

**Short term goal 8.** Classify “$J$-homogeneous” functors $\text{Top}_G^J \to \text{Top}_G^J$ for a finite $G$-set $J$.

**Progress.** The first step in Goodwillie’s classification is to show that $n$-homogeneous functors take value in infinite loop spaces. I proved the equivariant analogue of this result in [23]: “$J$-homogeneous” functors deloop by the permutation representation of $J$. Let us suppose for simplicity that $J = nG$ is free. Genuine $G$-spectra with a naïve $\Sigma_n$-action classify functors that are $nG$-excisive and $(n-1)$-reduced [23], whereas $nG$-homogeneous functors are by definition $nG$-excisive and $(n-1)G$-reduced. The calculation of the layers of the equivariant tower of the identity of [23] suggests that $nG$-homogeneous functors should be of the form

$$\Omega^n G((C \wedge X^\wedge n) \wedge \Sigma_n E\mathcal{F}_n)$$

where $\Omega^n G$ is the genuine infinite loop space functor, and $E\mathcal{F}_n$ is a classifying space for the family $\mathcal{F}_n$ of the subgroups of $G \times \Sigma_n$ whose quotients are $\Sigma_n$-free. Here $C$ is an “$\mathcal{F}_n$-spectrum”, a $G \times \Sigma_n$-spectrum whose homotopy type is determined by the fixed-points spectra for the subgroups that belong to $\mathcal{F}_n$. In the language of [5] this is a Mackey functor on the full subcategory of the orbit category of $G$ spanned by $\mathcal{F}_n$. It is still unclear what kind of family of subgroups should appear for a general $G$-set $J$. This classification is work in progress in a joint project with Tomer Schlank [27].

**Longer term goal 9.** Find a suitable notion of “equivariant analytic functor” that guarantees the convergence of the equivariant Goodwillie tower of a functor $F: \text{Top}_*^G \to \text{Top}_*^G$.

**Progress.** I proved in [23] that this convergence issue can be reduced to Goodwillie’s notion of analytic functors if $F$ “commutes with fixed-points”. This is for example the case for the identity functor, but not for the functors of Goal 7. This condition should be phrased in terms of equivariant connectivity estimates similar to the equivariant Blakers-Massey Theorem of [22].

Another important feature of Goodwillie calculus is the relationship with operads. The derivatives of the identity functor on pointed spaces have the structure of an operad in spectra, which is dual to the co-operad structure on the partition complexes, and whose homology is the Lie operad [18]. Moreover the chain rule in functor calculus equips the derivatives of a homotopy functor with the structure of a module over this operad [2]. The calculation of the equivariant derivatives of the identity on pointed $G$-spaces of [23] together with the conjectured classification of homogeneous functors of Goal 8 suggests that a similar chain rule should exist in the equivariant setting.

**Short term goal 10.** Develop a homotopy theory of “genuine” equivariant operads in an equivariant model category, and of algebras and modules over them.
Strategy. This is an ongoing project joint with Tomer Schlank [27]. Classically a $G$-operad, say in spaces, is simply defined as an operad in the category of $G$-spaces ([47] and [19]). Although these are the right objects, the levelwise equivalences of $G$-spaces produce a homotopy theory of “naïve” $G$-operads. The classification of Goal 8 suggests that an equivalence of $G$-operads should be defined by the fixed-points for the subgroups of $G \times \Sigma_n$ that belong to the families $F_n$. In this model structure the $N_\infty$-operads of [12], used to encode the multiplicative norms of $G$-commutative ring spectra of [40], will be a “$\Sigma$-cofibrant” replacement of the commutative operad. We are constructing these model structures on operads, algebras and modules by generalizing the transport techniques of [8] and [46].

Additionally, [7] contains a definition of $G$-operad (in the infinity categorical setting) in terms of multi-operations which are indexed on finite $G$-sets. We are in the process of proving that the two approaches are in fact equivalent.

The calculation of the equivariant derivatives of the identity of [23] shows that the equivariant derivatives of the identity functor of pointed $G$-spaces have the structure of a $G$-operad, induced by the co-operad structure on the partition complexes of [18].

Longer term goal 11. Prove a chain rule for the equivariant derivatives of homotopy functors, and relate the homology of the derivatives of the identity to an equivariant Lie operad.

Strategy. In joint work with Tomer Schlank [27]. The equivariant derivatives of a composition of functors should be equivalent to the composition product of symmetric sequences as in [2]. By “equivalent” we now mean in the sense of Goal 10, with respect to the subgroups of $G \times \Sigma_n$ which belong to the families $F_n$. It is still unclear if the homology of the derivatives of the identity correspond to a known algebraic structure of $G$-Lie operad.

References


