18.100B Problem Set 8

Due in class Monday, April 27. You may discuss the problems with other students, but you should write solutions entirely on your own.

1. This is a version of problems 8 and 10 in the text (pages 166-167). Let $S = \{x_1, x_2, \ldots \}$ be a countable subset of $(0,1)$. (For example, $S$ might consist of all the rational numbers between 0 and 1.) Define a real-valued function $f$ on $[0,1]$ by

$$f(x) = \sum_{\{n | x_n \leq x\}} 2^{-n}.$$  

The notation means that the sum extends exactly over those positive integers $n$ for which $x_n \leq x$.

a) Show that $f$ is a well-defined increasing function on $[0,1]$, that $f(0) = 0$, and $f(1) = 1$. (By Theorem 6.9, it follows that $f$ is Riemann-integrable.)

b) Show that $f$ is continuous at $x$ if and only if $x \notin S$.

c) Suppose that $x \notin S$. Prove that $f$ is differentiable at $x$.

d) Suppose $S$ is dense in $[0,1]$. Does the derivative $f'(x)$ exist for any value of $x$? (In this case part (c) doesn’t provide any places where the derivative exists. This question is quite a bit harder than any of the others; don’t worry if you can’t make any progress on it.)

2. Text, page 165, number 3.


4. Text, page 168, number 14. (If you’re artistically inclined, you might try defining $x_N(t)$ and $y_N(t)$ to be the $N$th partial sums of the series defining $x$ and $y$, and then try to sketch the parametric curves $\Phi_N(t) = (x_N(t), y_N(t))$ for $N = 1$ and 2.)