1. Homework 9

Problem. The Weyl algebra is the algebra of differential operators on $\mathbb{R}^n$ with polynomial coefficients. Consider the subalgebra $g$ of the Weyl algebra that’s generated by differential operators $\frac{\partial^2}{\partial x_i \partial x_j}$ and $x_i x_j$ for $1 \leq i \leq j \leq n$.

(1) Find a basis of $g$;
(2) Find a Cartan subalgebra $\mathfrak{h}$ and the root space decomposition of $g$;
(3) Choose a set of positive roots $R^+(g, \mathfrak{h})$ so that the nilradical $n$ contains $\frac{\partial^2}{\partial x_i \partial x_j}$ for all $1 \leq i \leq j \leq n$;
(4) The Weyl algebra acts on $V = \mathbb{C}[x_1, x_2, \ldots, x_n]$. When restricted to $g$, $V$ is an infinite dimensional representation of $g$. Prove that $V$ is a direct sum of two irreducible highest weight modules of $g$.

2. Homework 10

Problem. Suppose that $G$ is a compact Lie group and acts smoothly on a locally compact space $X$ and thus on $C(X)$. Suppose $(\pi_1, E_1)$ and $(\pi_2, E_2)$ are two irreducible representations and $\phi_i : E_i \to C(X)$ are injections respecting the $G$-action. Define $\phi_{E_1 \otimes E_2} : C(X) \to E_1 \otimes E_2 \mapsto \phi_1(e_1)\phi_2(e_2)$ using the universal property of the tensor product.

(1) Show that $\phi_{E_1 \otimes E_2}$ respects the $G$-action.
(2) Given any $E_1, E_2$, show that we can find $X$ and $\phi_i (i = 1, 2)$, such that $\phi_{E_1 \otimes E_2}$ is injective.
(3) Prove the equivalence of the following three statements:
(a) $(\pi, E)$ appears in $C(X)$
(b) There exists $x \in X$, such that $(\pi, E)$ appears in $C(G \cdot x)$.
(c) $\pi|_{G_x}$ contains the trivial representation of $G_x$ (the isotropy group of $G$ fixing $x$).