A standard way to make a group $G$ is to start with a large group $L$, list some elements $\{g_\alpha \mid \alpha \in A\}$ of $L$, and to define $G$ to be the smallest subgroup of $L$ containing all the elements $g_\alpha$. Notation is $G = \langle g_\alpha \mid \alpha \in A \rangle \subset L$.

The index notation is just a fancy way of allowing a finite or infinite set of indices. I might specify three elements $\{g_1, g_2, g_3\}$, in which case $A = \{1, 2, 3\}$; or I might specify a countably infinite set of elements $\{g_1, g_2, g_3, \cdots\}$, in which case $A = \mathbb{N}$; or I might specify one element for every real number $\{g_t \mid t \in \mathbb{R}\}$, in which case $A = \mathbb{R}$. The notation is flexible.

We say that we are defining $G$ by generators. The generators are the elements $g_\alpha$, and every element of $G$ can be written (not at all uniquely) as a word in the generators:

$$g = g_\alpha_1 g_\alpha_2 \cdots g_\alpha_r.$$  

Here the $\alpha_i$ are $r$ (not necessarily distinct) elements of $A$, and each exponent $\epsilon_i$ is $\pm 1$. We call this a word of length $r$ in the generators. The case $r = 0$ (the empty word) is allowed: it’s by definition the identity element of $L$. It’s very easy to check that this collection of words is indeed a subgroup of $L$, the smallest one containing all the generators.

An example of this is to take $L = GL(n, \mathbb{R})$ (invertible $n \times n$ matrices), and to define

$$A = \{(n - 1)\text{-dimensional subspaces } \alpha \subset \mathbb{R}^n\}.$$  

For each $\alpha \in A$ we define a linear transformation $g_\alpha$ by

$$g_\alpha(v) = \begin{cases} v & v \in \alpha \\ \cdot v & v \perp \alpha. \end{cases}$$  

This is the orthogonal reflection in the hyperplane $\alpha$. It turns out that

$$\langle g_\alpha \mid \alpha \text{ hyperplane in } \mathbb{R}^n \rangle = O(n),$$

the group of $n \times n$ orthogonal matrices. The example for this problem set is like that.

Define $V = C^\infty(\mathbb{R})$ to be the complex vector space of all infinitely differentiable complex-valued functions on the real line. The big group we will look at is $L = GL(V)$, all invertible linear transformations of the vector space of smooth functions (of a variable we’ll always call $x$).

Calculus is particularly concerned with three kinds of linear transformation:

$$(T_t f)(x) = f(x - t) \quad (t \in \mathbb{R}) \quad \text{left translation by } t$$  

$$(M_\xi f)(x) = \exp(-2\pi i \xi x) \quad (\xi \in \mathbb{R}) \quad \text{multiplication by } \exp(i \xi x)$$  

$$(Z_\theta f)(x) = \exp(2\pi i \theta) f \quad (\theta \in \mathbb{R}/\mathbb{Z}) \quad \text{scalar multiplication by } \exp(2\pi i \theta).$$

(1) Prove that $T_t T_s = T_{t+s}$. In particular (since $T_0$ is the identity), $T_t$ is invertible.

(2) Translation by $t$ is a one-parameter group of diffeomorphisms of $\mathbb{R}$. Find the corresponding vector field $X$ on $\mathbb{R}$.

(3) Prove that translation $T_t$ is characterized by a differential equation

$$\frac{d(T_t f)}{dt} = X(T_t f).$$  

(Each side of this equation is a map from $t \in \mathbb{R}$ to $V$.)

(4) Prove that $M_\xi M_\mu = M_{\xi + \mu}$. In particular, $M_\xi$ is invertible.

(5) Is there a differential equation like the one in (3) characterizing $M_\xi$?

(6) You may assume that $Z_\theta Z_\phi = Z_{\theta + \phi}$. Is there a differential equation characterizing $Z_\theta$?

(7) Define the Heisenberg group

$$G = \langle T_t, M_\xi, Z_\theta \mid t \in \mathbb{R}, \xi \in \mathbb{R}, \theta \in \mathbb{R}/\mathbb{Z} \rangle \subset GL(C^\infty(\mathbb{R})).$$  

Prove that every element $g \in G$ can be written uniquely as

$$g = Z_\theta M_\xi T_t \quad (t \in \mathbb{R}, \xi \in \mathbb{R}, \theta \in \mathbb{R}/\mathbb{Z}).$$

(8) Explain how to make $G$ a Lie group. Calculate $\pi_1(G)$.  